INTRODUCTION

Most of economic or financial decisions are such that they take place in an uncertain context. Taking a decision amounts to choose between several actions tha have uncertain consequences. When you wonder wether you take an umbrella, you face a decision problem with uncertainty, and the way you react to uncertain weather says something about your risk preferences. What will you do if the weather forcast calls for 10% of rain? And what about with 50%? Obviously, the anwer will not be the same for each individual. Modelling the individual attitude towards risk is a necessary condition if we want to understand and to predict the behaviour of insurance and finance markets. Attitude toward risk is the key-feature of the demand and supply of finacial and insurance products. In the huge majority of situations, investors, traders, managers, are risk averse end try to reduce the risk exposure by buying insurance or financial products. In this first chapter we want to give some intuitions, some qualitative hints that we want to model.

1 Intuitions

1.1 Une histoire d'oeufs et de paniers bien connue, sauf qu'il s'agit de bateaux (the diversification principle)

It is widely acknowledged that an important step has been done in the understanding of human behaviour in front of risk when Daniel Bernoulli (a famous swiss mathematician) wrote in 1738 a paper entitled "specimen theoriae novae de mensura sortis". In this paper Bernoulli tells the (fancied) story of a merchant, Sempronius, whose wealth is 4000 ducats at home, and, in addition, who owns 8000 ducats worth of goods and commodities in a foreign country, from where they can only be transported by sea. This transport is however risky (and that is the point that Bernoulli wants to develop) : the probability of a sinking during the trip is one half (these figures are not those taken by Bernoulli in the original paper, but they are easier to manipulate for our purpose).

Sempronius can choose to put the whole foreign wealth in one boat. In some sense he plays a lottery : with a probability 1/2 he will have 8000 ducats and with a probability 1/2, the boat is sunk and his wealth reduces to 4000 ducats.

He the has an ingenious idea : instead of trusting all his 4000 foreign ducats on one ship he splits them in equal proportions and trusts them on two ships! If we suppose that the two boats follow independant but equally dangerous routes, the new lottery he plays is then the following : with a probability 1/4 both ships arrive on port and his wealth will be 8000 ducats, with a probability 1/2 one of the two boats perish and his wealth is 6000 ducats and with probability 1/4 both ships are sunk and the wealth is 4000 ducats.

Bernoulli says that common wisdom suggests that the second lottery is strictly prefered : the value attached to the so called "diversified lottery" must be higher. As (as it is easy to see,) the two lotteries have the same expectation (6000 ducats), Expectation is not an appropriate index to measure the value of a lottery for a "human" decision maker.

What is hence the difference between the two lotteries. It is easy to sea that the second one is less risky in the sense that extremal events (4000 and 8000) have less "weight" (to the profit of central event 6000).

The conclusion here is that if we want to have a model whichs describes accurately this type of behaviour we must find a comparision criterion which valuates such reduction of risk.

We want a model which follows the diversification principle.

1.2 Un tiens vaut mieux que deux tu l'auras (risk aversion)

Let's play. I propose two very simple lotteries. In the first one: I flip a coin and if it falls heads you win 2 euros, if it is tails, sorry, but you win nothing. In the second one whatever the toss you win β euros! The followins table gives a simple description of the two lotteries.

	1/2	1/2
lottery 1	2	0
lottery 2	β	β

Which lottery do you prefer? It obviuously depends on the value of β! Indeed, if $\beta = 0$ you certainly prefer the lottery 1. and if $\beta = 2$ you certainly prefer the lottery 2. It must hence exists a value $\widehat{\beta}$ where the preference switches. But this threshold is likely to be different for different people! Let's keep on. What do you think of an individual whose $\widehat{\beta} \geq 1$. He is ready to take risk more frequently : he is a risk lover. If $\widehat{\beta}$ is very close to 2 we can say that he loves risk almost infinitely. Conversely, if $\widehat{\beta} \leq 1$ the player prefers "sures" situations : he is risk averse, infinitely risk averse if $\widehat{\beta}$ is close to 0.

In the sequel we will suppose that the prevalent behaviour is risk aversion. We will suppose that decsion makers are risk averse.

1.3 Quand le bonheur des uns fait le malheur des autres (et réciproquement) mutualization principle.

On a exotic island in the south pacific ocean, the weather regime is quite special. When it is sunny on the West coast, it is raining (cats and dogs) on the *East* coast. And this lasts for the whole touristic season. Obviously the weather is uncertain in the sense that before the beginning of the season there is equal probability of sun and rain.

When it is sunny on the West coast, the touristic resort of the West coast is flourishing (profit amount to 2M$) although, the one of the East coast doesn't make money (0M$). And conversely when it is sunny on the East coast.

From the point of view of both resorts, profit is a random variable : with probability 1/2 the wealth will be 2M$ and with probablity 1/2, it will be equal to 0M$.

If there is an hermetic boudary between the two regions, each of them supports a risk : the wealth is a random variable. It is the lottery whe have proposed in the previous paragraph.

But, as a finance expert, or an insurer, you can have an ingenious idea.

Suppose that the two resort's owners are risk-averse. We have seen that this means that they are likey to prefer a lottery in which they win surely $\widehat{\beta}$ M$ with $\widehat{\beta} = 1 - \pi$, where π is ≥ 0!

There is a trick that can be beneficial for both resort's owners and you (as a financial or insurance expert). You have remarked that the TOTAL wealth is not risky at all : it amounts exactly to 2M$.

These 2M$ (sure) can be splited in three parts : $1-\pi$ for the W resort, $1-\pi$ for the East resort and 2π for you!

How can you do this? You have several possibilities : you can propose an insurance contract. Each resort pays an insurance premium $1+\pi$. If the weather is bad, the insurer pays an indemnity equal to $2.You$ can also propose to introduce each resort in the stock market. The West coast owner will hence buy 1/2 of the stocks of the East and reciprocally! (and you take a commission π)

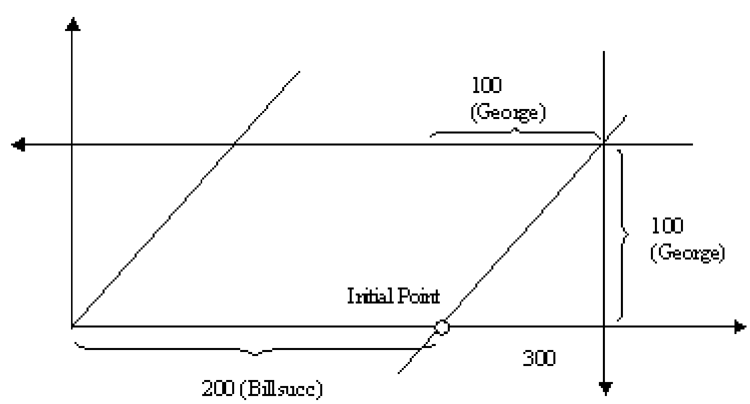
1.4 What is the price of uncertainty?

The previous example show that if people are risk averse, they are ready to pay to get rid of risk, or more subtly, they accept to take risk if the price to pay is sufficiently low compared to the benefits. It hence follows from the assumption of risk aversion that the price of a lottery must be lower than the expectation of its payments. But this price also of the "supply and demand" law. What can we say about that?

Suppose two people. Bill is a young Engeneer or Searcher who has a project. His idea, according to him is very skilful are is likely to make a lot of money. In fact with a probability of 1/2 he will make 200 k$ and with a probability of 1/2 he will make 0!

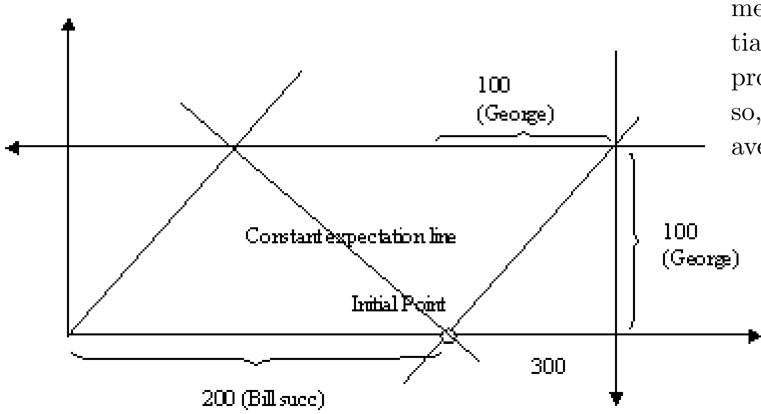
Obviuously, Bill does not like to take risk. He would like to find someone to share with him. But we are not in the same situation as the previous one where there was a natural partner with whom sharing risk was a very powerful trick since it led to cancel risk!

George is a retired guy whose wealth is not risky : he owns 100 k$ for sure. In this littlle economy the TOTAL wealth (unlike the previous story) is random : with a probability of 1/2 the TOTAL wealth is 300 and with a probability 1/2 it amounts to 100. On the following picture, on the vertical axis we represent the wealth when Bill's project fails and on the horizontal axis when it succeeds.



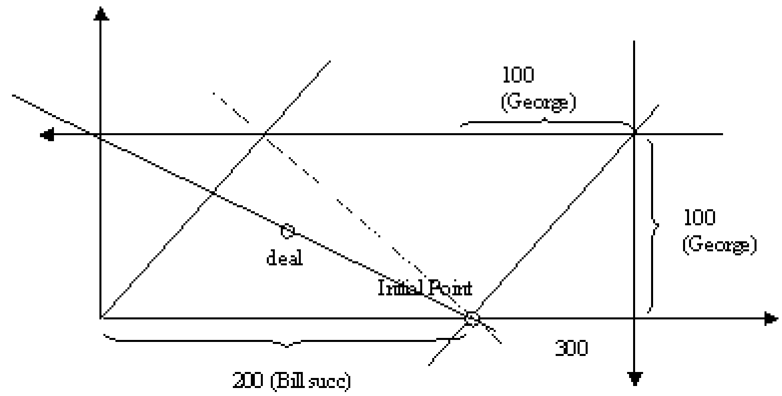
The point (200,0) for Bill and (100,100) for George depicts the initial situation. Every point in the rectangle 300,100 is a possible deal! The two 45° dashed lines are deals where there is no risk either for Bill (the first one) or George (the second one). We see immediately that (unlike the previous stoty) there is no deal that make the risk vanish for both people. Finding deals that are acceptable by both Bill and George is hence less easy. Let first depict deals that give lotteries which give the same expectation (for both Bill and Maurice) as the initial lottery. These are point (x, y) such that :

$$\frac{1}{2}x + \frac{1}{2}y = 100 \text{ for Bill}$$
$$\frac{1}{2}(300 - x) + \frac{1}{2}(100 - y) = 100$$
$$\text{which gives } \frac{1}{2}x + \frac{1}{2}y = 100 \text{ for George}$$



Suppose that both are risk averse. George accepts to take risk (that is to depart from his 45° line) if the deal is such that his expected income is increased, that is if the deal lies below the constant expectation line. We can guess that he will ask for a large risk "premium" if the deal is far from his 45° line. Bill would like to reduce his risk, that is to get closer to his 45° line. He accepts to pay for that and to ontain a deal that gives an expectation which is lower, that is below the constant expectation line.

To summarize, The accepted deal is likely to be as the one depicted on the following picture.



I have drawn a line betwwen the initial point and the deal. This line has an equation : $\pi x + (1 - \pi)y = 200\pi$. The previous reasoning tells us that $\pi < 1/2$. π is called "the risk neutral probability". With this probability measure, the deal has the same expectation as the initial point (that is the famous martingale property). This probability measure overweights bad events. By doing so, risk with the same expectation are acceptable by risk averse people.