CHAPTER 5

Continuous models

We want to replace the "tree model" of chapter 3 by some continuous process. The idea is very simple. between t and $t+dt$ instead of having a finite set of possibilities, we want to have an "infinite" non countable set of "states". We would like to define a random continuous variable "drawn" at each date in each previous state. There is a way to do that : brownian motions.

5.1. Brownian motion

DEFINITION 41. A standard Brownian Motion (SBM) is a process $B(t)$ such that :

1. $B(0) = 0$
2. B has a continuous path
3. for all subdivision $t_0 = 0$, $t_1, ..., t_k, t_n = T$, $B(t_{k+1}) - B(t_k)$ and $B(t_{h+1}) - B(t_h)$ are independant
4. $B(t_{k+1}) - B(t_k)$ is a normal variable with zero mean and varaiance equal to $t_{k+1} - t_k$.

The idea above is very simple. First this definition is consistent because, and it is very important, normal distribution is stable : if two variables are normal, then any linear combination is normal. So the definition above does not depend on the way you subdivise the interval.

A brownian motion is a generalization of a random walk : at each "step" B "moves" randomly (increases or decreases).

Let us examine the main properties of a SBM.

5.1.1. Quadratic variations.

Let $B(t)$ a SBM : $B(t') - B(t)$ is a normal variable centered with variance $t' - t$.

Let one interval $[0, T]$ and a subdivision t_i.

We want to study :

$$\Delta_k = (B(t_{k+1}) - B(t_k))^2 - (t_{k+1} - t_k)$$

We have :

$$\sum \Delta_k = \sum (B(t_{k+1}) - B(t_k))^2 - T$$

Let us examine the moments of Δ_k.

First, the mean is zero since $B(t_{k+1}) - B(t_k)$ is precisely a normal variable with variance $(t_{k+1} - t_k)$,

$$\mathrm{E}\left[\Delta_k\right] = \mathrm{E}\left[(B(t_{k+1}) - B(t_k))^{\,2}\right] - (t_{k+1} - t_k)$$

$$= \mathrm{var}\left[B(t_{k+1}) - B(t_k)\right] - (t_{k+1} - t_k) = 0$$

What is its variance (remenber that the fourth moment of a normal is $3\sigma^4$) :

$$\begin{aligned}
\mathrm{var}(\Delta_k) &= E((B(t_{k+1}) - B(t_k))^4 - 2(t_{k+1} - t_k)(B(t_{k+1}) - B(t_k))^2 + (t_{k+1} - t_k)^2) \\
&= 3\left(\mathrm{var}\left[B(t_{k+1}) - B(t_k)\right]\right)^2 - 2(t_{k+1} - t_k)\mathrm{var}\left[B(t_{k+1}) - B(t_k)\right] + (t_{k+1} - t_k)^2 \\
&= 2(t_{k+1} - t_k)^2
\end{aligned}$$

Take for example $t_{k+1} - t_k = 1/n$,

Then we have :

$$\lim \mathrm{var}(\sum \Delta_k) = 0$$

The variance of the quadratic variation is zero! That means that $\sum (B(t_{k+1}) - B(t_k))^2$ is not very far from T.

In fact one can show that : $\sum (B(t_{k+1}) - B(t_k))^2$ is asymptotically almost surely equal to T.

That allows to note :

$$[dB(t)]^2 = dt$$

This is obviuously an abuse of notation, but very useful...

5.2. Stochastic integral

Let $f(t)$ a real function of t. Just define :

$$\sum_{0}^{n-1} f(t_k)\left(B(t_{k+1}) - B(t_k)\right)$$

This is a normal variable whose mean is zero and whose variance is :

$$\sum_{0}^{n-1} (f(t_k))^2 (t_{k+1} - t_k)$$

What is the limit of the above sum? Obviously $\int_0^T (f(t))^2 \, dt$

That alllows us to define the stochastic integral of f :

$$X(T) = \int_0^T f(t) dB(t)$$

This is a normal variable whose mean is zero and variance $\int_0^T (f(t))^2 \, dt$.

More generally one can show that one can define the stochastic integral of a random f.

- Stochastic differential

Instead of $X(T) = \int_0^T f(t) dB(t)$ one may writes :

$$dX(t) = f(t) dB(t)$$

This extends the notion of differential, but the right writing would be under the integral form.

5.2.1. Stochastic differential equation. This allows us to define a kind of "stochastic differential equation".

One writes for instance :

$$dX(t) = \mu(X(t), t) dt + \sigma(X(t), t) dB(t)$$

This means that the random process X must be such that :

$$X(t) = X(0) + \int_0^t \mu(X(s), s) ds + \int_0^t \sigma(X(s), s) dB(s)$$

The second integral being a "stochastic integral".

The solution of such a "stochastic differentiel equation" is called a diffusion process.

A good example is the geometric brownian motion :

$$dX(t) = \mu X(t) dt + \sigma X(t) dB(t)$$

5.2.2. Martingale. In this context what is a martingale? Remember that in the discrete finite model, a random process is a martingale if the value at time t is equal to the mean of the values at the successors.

Here it can be shown that $X(t)$ is a martingale if there exists a process $\gamma(t)$ such that $dX(t) = \gamma(t)dB(t)$ where B is a standard Brownian.

5.3. Ito lemma

Let f a function of a real variable. We would like to study the process $f(B(t))$ where B is a SBM. Or more generally $f(X(t))$ where $X(t)$ is a diffusion process.

Just write :

$$f(B(t_{k+1}))-f(B(t_k)) = f'(B(t_k))\left(B(t_{k+1}) - B(t_k)\right)+\frac{1}{2}f''(B(t_k))\left(B(t_{k+1}) - B(t_k)\right)^2+R\left(\left(B(t_{k+1}) - B(t_k)\right)\right)$$

take the sum :

$$f(B(T)) - f(B(0)) = \sum f(B(t_{k+1})) - f(B(t_k))$$

The first and the second term give (at the limit) :

$$\int_0^T f(B(t))dB(t) + \frac{1}{2}\int_0^T f''(B(t))dt$$

on can show that the last term is almost surely 0 at the limit.

We can hence write (simple Ito lemma):

$$df(B(t)) = f'((B(t))dB(t) + \frac{1}{2}f''(B(t))dt$$

This formula is very easy to understand. Think of a particle moving on the real line according to the Brownian motion. Take for example $f(x) = x^2$, $f(B(t))$ is simply the square of the "distance" traveled by the particle. Obviously this distance is not zero on average. At each step the particle moves away by a distance equally distributed around a positive value equal to the time spent : $dX^2 = dt + 2B(t)dB(t)$.

More generally if we have a diffusion process :

$$dX(t) = \mu(t)dt + \sigma(t)dB(t)$$

And a real function $f(x,t)$.If we set $Y(t) = f(X(t),t)$. We can write :

$$dY(t) = \frac{\partial f}{\partial t}(X(t),t)dt + \frac{\partial f}{\partial x}(X(t),t)\left(\mu(t)dt + \sigma(t)dB(t)\right) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t),t)\left(\sigma(t)\right)^2 dt$$

Or :

LEMMA 42. *(Ito) Let $X(t)$ a diffusion process such that $dX(t) = \mu(t)dt + \sigma(t)dB(t)$. Let $Y(t) = f(X(t), t)$, where f is a C2 real function. Then Y is a diffusion process and :*

$$dY(t) = \left\{ \frac{\partial f}{\partial t}(X(t), t) + \frac{\partial f}{\partial x}(X(t), t)\mu(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t), t)\left(\sigma(t)\right)^2 \right\} dt + \frac{\partial f}{\partial x}(X(t), t)\sigma(t)dB(t)$$

We define the Dynkin operator by

$$D(X(t)) = \frac{\partial f}{\partial t}(X(t), t) + \frac{\partial f}{\partial x}(X(t), t)\mu(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X(t), t)\left(\sigma(t)\right)^2$$

This is simply the infinitesimal variation of the mean of the process Y.

Just look at a geometric brownian motion :

$$dX(t) = \mu X(t)dt + \sigma X(t)dB(t)$$

Take $Y(t) = \ln(X(t))$. By Ito formula we get :

$$dY(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dB(t)$$

That is :

$$X(t) = X(0) \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B(t)\right]$$

5.4. Continuous asset valuation

As in the discrete case, one can show that the no arbitrage condition amount to the existence of a "risk-free" probability. The price of an asset is in this case equal to the discounted expected value of the future cash flows. But first consider the deterministic case.

5.4.1. Deterministic continuous model.

Consider first a continuous deterministic model. Consider a zero-coupon bond of date of maturity T. Its value at date t is $B(t,T)$. The no arbitrage condition means that its yield between t and $t+h$, $\frac{B(t+h,T)-B(t,T)}{B(t,T)}$ does not depend on T. The limit when h goes to zero is called the short term interest rate $r(t)$.

PROPOSITION 43. *If $r(t)$ is the deterministic short term interest rate, we have $B(T,T) = 1$ and :*

$$\dot{B}(t,T) = r(t)B(t,T)$$

That is :

$$B(t,T) = \exp\left[-\int_t^T r(s)ds\right]$$

If one knows the dynamics of short term rate $r(t)$, then one knows all the zero coupon prices.

5.4.1.1. *The yield curve.* On can rewrite the price of the z-c bond introducing its yield as a function of maturity :

$$B(t,T) = \exp\left[-\int_t^T r(s)ds\right] = \exp\left[-R(t,T)(T-t)\right]$$

That is :

$$R(t,T) = \frac{\int_t^T r(s)ds}{T-t}$$

Writing it as a function of maturity τ:

$$\widehat{R}(t,\tau) = \frac{\int_t^{t+\tau} r(s)ds}{\tau}$$

Definition 44. $\tau \to \widehat{R}(t,\tau)$ is the "yield curve" at date $t:$ the yield per unit of time as a function of maturity.

5.4.1.2. *Pricing of more complicated assets.* Just first recall some very simple results, in continuous time, when there is no uncertainty.

Consider one asset A that gives a continuous cash flow $b(t)dt$.

Assume as above that the instantaneous yield is $r(t)dt$, or alternatively, that all the zero coupons are available.

With one euro at date t one has two possible strategies : buy the asset, earn the cash flow and resell it at date $t+dt$. or buy a zero coupon and get $1+r(t)dt$ per euro. Let $y(t)$ the value of asset A at date t. Arbitrage-free hypothesis implies :

$$\overset{\bullet}{y} + b = ry$$

It is quite easy to solve this differential equation :

Set

$$z(t) = z(t_0)\exp\left(\int_{t_0}^t r(s)ds\right)$$

and,

$$xz = y$$

we have hence

$$\overset{\bullet}{y} = \overset{\bullet}{x}z + x\overset{\bullet}{z} = \overset{\bullet}{x}z + ay = \overset{\bullet}{x}z + \overset{\bullet}{y} + b$$

and then :

$$\overset{\bullet}{x} = \frac{-b}{z}$$

or

$$x(t) = x(t_0) - \frac{1}{z(t_0)} \int_{t_0}^{t} \left\{ b(s) \exp\left(- \int_{t_0}^{s} r(u)du \right) \right\} ds$$

Hence :

$$y(t) = \exp\left(\int_{t_0}^{t} r(s)ds \right) \left[y(t_0) - \int_{t_0}^{t} b(s) \exp\left(- \int_{t_0}^{s} r(u)du \right) ds \right]$$

Or

$$y(t_0) = y(t) \exp\left(- \int_{t_0}^{t} r(s)ds \right) + \int_{t_0}^{t} b(s) \exp\left(- \int_{t_0}^{s} r(u)du \right) ds$$

The value at date t_0 is simply the discounted value of future earnings. The discount factor is simply $\exp\left(- \int_{t_0}^{s} r(u)du \right)$ that is the value of the z-c with maturity $s - t_0$.

$$y(t_0) = y(t)B\left(t_0, t\right) + \int_{t_0}^{t} b(s)B\left(t_0, s\right) ds$$

5.4.1.3. *Examples.* The above formula has two components : the first one takes into account the resell value at date t. The second is the present value of the cash flow.

Consider then the following particular cases :

(1) Finite bond. Imagine there is a time T such that $b(t) = 0$ for $t \geq T$. This implies $y(T) = 0$, so that:

$$y(t_0) = \int_{t_0}^{T} b(s) \exp\left(- \int_{t_0}^{s} r(u)du \right) ds$$

that is the discounted value of cash flow: the fundamental

(2) Infinite living asset with no dividend: pure bubble :

$$y(t) = y(0) \exp\left(\int_{0}^{t} r(s)ds \right)$$

here the value grows at a rate equal to r

(3) General case : fondamental and bubble part. Let $f(t) = \int_t^\infty b(s) \exp\left(-\int_t^s r(u)du\right) ds$, the "fondamental value" of the asset, we have : $\overset{\bullet}{f} = b + rf$, so that

$$\overset{\bullet}{y - f} = r(y - f)$$

$y - f$ is a pure bubble

These examples shows that arbitrage free condition does not prevent bubbles (permanent bubbles) growing at the rate r.

5.4.2. derivatives. Imagine now there is a "state variable" governed by the following equation :

$$\overset{\bullet}{X}(t) = f(t, X(t))$$

Let $s \to \widehat{X}(t, x, s)$ the trajectory, that is the value of X at date s when the value at t is x. To be simple we write, when there is no ambiguity :

$$x(s) = \widehat{X}(t, x, s)$$

We have an asset (kind of) "derivative" whose cash flow $b(t, x)$ depends on the value of the state variable, and that terminates at T giving $h(x)$ en T.

One assumes that the yield of the risk-free asset depends on the state variable: $r(t, x)$

Let $V(t, x)$ the value of the asset , $v(s) = V(s, X(t, x, s))$

Hence :

$$\overset{\bullet}{v}(s) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(s, x(s))$$

Or :

$$\overset{\bullet}{v}(s) + b(s, x(s)) = r(s, x(s))v(s)$$

So :

$$V(t, x) = v(t) = h(x(T)) \exp\left(-\int_t^T r(s, x(s))ds\right)$$

$$+ \int_t^T b(s, x(s)) \exp\left(-\int_t^s r(u, x(u))du\right) ds$$

5.4.3. Stochastic models. Introduce now randomness. We are going to study very simple models. The natural way is to assume that there exists a stock whose value at date t is $S(t)$. We assume that $S(t)$ is a diffusion process. This stock does not distribute dividends. We assume moreover there exists a risk free asset whose instantaneous yield is $r(t)dt$. Let $\widehat{S}(t) = \exp\left[-\int_0^t r(u)du\right] S(t)$ the discounted value of the stock at 0.

5.4.3.1. *Direct approach.* In the discrete case we have seen that the discounted value of an asset distributing no dividend was a martingale for "some" probability distribution. The (discounted) value of any other asset will then be the expected (with this probability) discounted value of future earnings. There is a "common" probability distribution such that the value of any asset is simply equal to the expected discounted value of its cash flows.

We hence need a probability structure such that $\widehat{S}$ is a martingale. If this is the case, (see above) there exists a SBM such that :

$$d\widehat{S}(t) = \sigma(t, S(t))dB(t)$$

Then necessarily:

$$dS(t) = r(t)S(t)dt + \sigma(t, S(t))dB(t)$$

Consider any portfolio (with stock and the risk free asset) whose value at t is $V(t)$.

Consider the following strategy : at t one sells this portfolio and buy $\delta(t)$ stocks. The variation of the value between $t$and $t + dt$ is :

$$dV(t) = (V(t) - \delta(t)S(t))r(t)dt + \delta(t)dS(t)$$

That is :

$$dV(t) = V(t)r(t)dt + \delta(t)\left(dS(t) - r(t)S(t)dt\right)$$

Or :

$$dV(t) = V(t)r(t)dt + \delta(t)\sigma(t, S(t))dB(t)$$

If we note $\widehat{V}$ the discounted value of the portfolio :

$$d\widehat{V}(t) = \exp\left(-\int_0^t r(s)ds\right)\delta(t)\sigma(t, S(t))dB(t)$$

Which means that $\widehat{V}$ is a martingale.

5.4.3.2. *Indirect approach.* Instead of defining a priori the risk-free probability, assume now that the process $S(t)$ is given by :

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dB(t)$$

Consider a portfolio whose value is $X(t)$, with a strategy $\delta(t)$:

$$dX(t) = r(t)X(t)dt + \delta(t)S(t)\sigma(t)\left[\frac{\mu(t) - r(t)}{\sigma(t)}dt + dB(t)\right]$$

If the second term where a "dB" $\widehat{X}$ would be a martingale. But it is not! We need to "distort" the initial probability structure such that this term can be considered as a brownian term.

Let $\theta(t) = \frac{\mu(t)-r(t)}{\sigma(t)}$ and $d\widetilde{B}(t) = \theta(t)dt + dB(t)$.

Consider now the random variable $Z(t) = \exp\left[Y(t)\right]$ with $dY(t) = -\frac{1}{2}\left(\theta\left(t\right)\right)^2 dt - \theta(t)dB(t)$, with $Y(0) = 0$.

By Ito : $dZ(t) = \exp\left[Y(t)\right]\left(dY(t) + \frac{1}{2}\left(\theta\left(t\right)\right)^2 dt\right) = -\theta(t)Z(t)dB(t)$.

This means that $Z(t)$ is a martingale such that $\mathrm{E}(Z(t)) = 1$.

We have :

$$Z(t) = \exp\left[-\frac{1}{2}\int_0^t \left(\theta\left(t\right)\right)^2 dt - \int_0^t \theta(t)dB(t)\right]$$

And :

$$d\widehat{S}(t) = \left(\mu(t) - r(t)\right)\widehat{S}(t)dt + \sigma(t)\widehat{S}(t)dB(t)$$

That is :

$$\widehat{S}(t) = \widehat{S}(0)\exp\left[\int_0^t \sigma(t)\left(\theta(t) - \frac{1}{2}\sigma(t)\right)dt + \int_0^t \sigma(t)dB(t)\right]$$

Multiply by Z:

$$Z\hat{S} = \widehat{S}(0)\exp\left[\int_0^t \left(\sigma(t)\theta(t) - \frac{1}{2}\left(\sigma(t)\right)^2 - \frac{1}{2}\left(\theta\left(t\right)\right)^2\right)dt + \int_0^t \left(\sigma(t) - \theta(t)\right)dB(t)\right]$$

$$Z\hat{S} = \widehat{S}(0)\exp\left[-\frac{1}{2}\int_0^t \left(\sigma(t) - \theta(t)\right)^2 dt + \int_0^t \left(\sigma(t) - \theta(t)\right)dB(t)\right]$$

Which means :

$$d\left(Z(t)\widehat{S}(t)\right) = \left(\sigma(t) - \theta(t)\right)Z(t)\widehat{S}(t)dB(t)$$

When we multiply the (discounted) values by $Z(t)$ we obtain a martingale.

The idea is quite simple : the initial brownian has a probability density :

$$f(t,b) = \frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{b^2}{2t^2}\right]$$

Take a "particular" path $B(t,\omega)$ that is a possible trajectory of the brownian, at each date the value $B(t,\omega)$ is weighted by $\frac{1}{\sqrt{2\pi t}}\exp\left[-\frac{B(t,\omega)^2}{2t^2}\right]$. We can reweight it by multiplying it by the value taken by $Z(t,\omega)$. This will be an acceptable reweighting since the sum through all the possible trajectories will be 1 because $\mathrm{E}\left[Z(t)\right]=1$.

PROPOSITION 45. *If $X(t)$ is the value of any portfolio, then $Z(t)\widehat{X}(t)$ is a martingale*

5.4.4. Black et Scholes. Here we simply assume that r and σ are constant. The direct approach gives the "risk neutral dynamics" :

$$dS = rSdt + \sigma SdB$$
$$S(T) = \exp\left[\ln(S) + \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(B(T) - B(t)\right)\right]$$

For a derivative without dividend but giving $h(S(T))$ à T, the price $P(t,S)$ will be:

$$P(t,S) = \exp(-r\,(T-t))\mathrm{E}\left[h(S(T))/S(t) = S\right]$$

for a Call $h(S(T)) = (S(T)-K)^+$

$$C(t,S) = \exp(-r\,(T-t))\mathrm{E}\left[(S(T)-K)^+/S(t) = S\right]$$

This is easily computable bu the mean of some lemmas :

LEMMA 46. *Let a gaussian variable with mean μ and variance s^2 . Compute $I = E\left[(exp(u)-K)^+\right]$*

$$I = \frac{1}{\sqrt{2\pi}s}\int_{ln(K)}^{+\infty}\left(e^u - K\right)e^{-\frac{(u-\mu)^2}{2s^2}}du$$

PROOF. Set $t = \frac{u-\mu}{s}$:

$$I = \frac{1}{\sqrt{2\pi}}\int_{\frac{\ln(K)-\mu}{s}}^{+\infty}\left(e^{\frac{2st+2\mu-t^2}{2}} - Ke^{-\frac{t^2}{2}}\right)dt$$
$$I = \frac{1}{\sqrt{2\pi}}\int_{\frac{\ln(K)-\mu}{s}}^{+\infty}\left(e^{-\frac{(t-s)^2-2\mu-s^2}{2}} - Ke^{-\frac{t^2}{2}}\right)dt$$

Or : □

$$I = \frac{1}{\sqrt{2\pi}} e^{\mu + \frac{s^2}{2}} \int_{\frac{\ln(K)-\mu}{s}}^{+\infty} e^{-\frac{(t-s)^2}{2}} - \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln(K)-\mu}{s}}^{+\infty} K e^{-\frac{t^2}{2}} dt$$

That is :

$$I = e^{\mu + \frac{s^2}{2}} \overline{N} \left(\frac{\ln(K) - \mu}{s} - s \right) - K \overline{N} \left(\frac{\ln(K) - \mu}{s} \right)$$

Where $\overline{N}$ is the decumulative of the normal $\overline{N}(x) = \int_x^{+\infty} e^{-\frac{u^2)}{2}} du$.

We have then the Black and Scholes formula by taking :

$$\mu = \ln(S) + \left(r - \frac{1}{2}\sigma^2 \right)(T - t)$$

$$s = \sigma\sqrt{T - t}$$

in the expression :

$$C(t, S) = \exp(-r(T - t))\mathrm{E}\left[(S(T) - K)^+ / S(t) = S \right]$$