CHAPTER 4

Microstructure and behaviour models

We depart here from the hypothesis of "always rational agents" and perfect markets.

4.1. The market efficiency hypothesis

The concept of efficiency for financial market corresponds to the idea that the current prices of financial assets perfecly reflect all the information available to investors. In other words, financial markets are efficient when an informed trader cannot make systematically more money than other who only observe current prices.

There are three different degrees of market efficiency. To explain this, take the example of one asset (distributing no dividends) whose price is p_t at date t. At date t the price p_{t+1} is "random".

The weakest notion of market efficiency says that price is a a "martingale" for some probability distribution :

$$p_t = \mathbb{E}\left(p_{t+1}/p_t\right)$$

The semi-strong efficiency says that the price at date t reflects all public information of date t:

$$p_t = \mathbb{E}\left(p_{t+1}/\mathcal{I}_t\right)$$

The strong efficiency assumption says that the price reflects all public and private information : the equilibrium process makes public all private information.

To precise these concepts we are going to study one very simple model proposed by Grosman and Stiglitz.

4.2. The Competitive REE

There is a single, risky asset with random liquidation value $\widetilde{\theta}$ and riskless asset (with unitary return). These are traded by risk averse agents and "noise traders." The utility derived by a trader i for the (random) profit $\pi_i = (\widetilde{\theta} - p)x_i$ of buying x_i units of the asset at price p is of the CARA type: $U(\pi_i) = -\exp -\rho_i \pi_i$, where$\rho_i > 0$ is the CARA coefficient that measures risk aversion. We call risk tolerance the inverse of risk aversion $t_i = \frac{1}{\rho_i}$. Initial wealth of each trader i is normalized to 0 (wlog). Trader i is endowed with a piece of private information about $\widetilde{\theta}$. Noise traders are assumed to trade for liquidity reasons submitting a random trade $\widetilde{u}$.

Suppose that a fraction of traders $\mu \in [0,1]$ receives a private signal $\widetilde{s}$ about $\widetilde{\theta}$, we call them Informed, subscript I, while the complementary fraction does not, Uninformed, subscript U . Both classes of traders condition their orders on the price p. Let $\rho_i = \rho_I > 0$ for Informed and $\rho_i = \rho_U \geq 0$, for Uninformed. $\widetilde{s}, \widetilde{\varepsilon}, \widetilde{u}$ are (pairwise independent) normally distributed:

$$\widetilde{s} \rightsquigarrow N(\overline{\theta}, \sigma_s^2)$$

$$\widetilde{\theta} = \widetilde{s} + \widetilde{\varepsilon} \quad , \widetilde{\varepsilon} \rightsquigarrow N(0, \sigma_\varepsilon^2)$$
$$\widetilde{u} \rightsquigarrow N(0, \sigma_u^2)$$

We call "precision" the inverse of the variance for $j = s, \varepsilon$ or u :

$$\tau_j = \frac{1}{\sigma_j^2}$$

If the price is p , what is the demand of a trader?

The quatity demanded maximizes the expected utility, the expectation being conditionnal to the information $\mathcal{J}$ detained :

$$X_i(p/J) = \arg\max \left(\mathbb{E}\left[U_i\left(\left(\widetilde{\theta} - p \right) x_i \right) / \mathcal{J} \right] \right)$$

But we know that $\widetilde{\theta}$ is normally distributed so that :

$$\mathbb{E}\left[U_i\left(\left(\widetilde{\theta} - p \right) x_i \right) / \mathcal{J} \right] = U_i\left(\mathbb{E}\left[\left(\widetilde{\theta} - p \right) x_i / \mathcal{J} \right] - \frac{1}{2}\rho_i \text{var}\left(\left(\widetilde{\theta} - p \right) x_i / \mathcal{J} \right) \right)$$

So that i's maximization amounts to :

$$\max \left\{ \mathbb{E}\left[\left(\widetilde{\theta} - p \right) x_i / \mathcal{J} \right] - \frac{1}{2}\rho_i \text{var}\left(\left(\widetilde{\theta} - p \right) x_i / \mathcal{J} \right) \right\}$$

which gives :

$$X_i(p/\mathcal{J}) = \frac{\mathbb{E}\left[\widetilde{\theta} / \mathcal{J} \right] - p}{\rho_i \text{var}\left(\widetilde{\theta} / \mathcal{J} \right)}$$

This expression is quite intuitive : the demand is proportionnal to the spread between expected value and price. The coefficient of proportionality is large when risk aversion is low and/or risk (measured through variance) is low.

Before computing the equilibrium we must recall some simple formulas when random variables are normal.

Let $\widetilde{x}$,$\widetilde{y}$ a gaussian vector (i.e the variables are such that every linear combination is gaussian).

we have :

LEMMA. *$\widetilde{x}$,$\widetilde{y}$ a gaussian vector, we have :*

$$\mathbb{E}\left(\widetilde{x}/\widetilde{y}\right)-\mathbb{E}\left(\widetilde{x}\right)=\frac{cov\left(\widetilde{x},\widetilde{y}\right)}{var\ \left(\widetilde{y}\right)}\left[\widetilde{y}-\mathbb{E}\left(\widetilde{y}\right)\right]$$

$$var\ \left(\widetilde{x}/\widetilde{y}\right)=var\left(\widetilde{x}-\mathbb{E}\left(\widetilde{x}/\widetilde{y}\right)\right)=var\left(\widetilde{x}\right)-\frac{\left(cov\left(\widetilde{x},\widetilde{y}\right)\right)^{2}}{var\left(\widetilde{y}\right)}=\left(1-\frac{\left(cov\left(\widetilde{x},\widetilde{y}\right)\right)^{2}}{var\left(\widetilde{x}\right)var\left(\widetilde{y}\right)}\right)var\left(\widetilde{x}\right)$$

4.2.1. Naïve equilibrium. Now, consider a first "naive" equilibrium. Each trader optimize with his information. Uninformed has no information (noted $\mathcal{J}_U^0$) so that $\mathbb{E}\left[\widetilde{\theta}/\mathcal{J}_U^0\right]=\overline{s}$ and var $\left(\widetilde{\theta}/\mathcal{J}_U^0\right)=\sigma_s^2+\sigma_\varepsilon^2$:

$$X_U(p/\mathcal{J}_U^0)=\frac{\overline{\theta}-p}{\rho_U\left(\sigma_s^2+\sigma_\varepsilon^2\right)}=t_U\tau_\theta\left(\overline{\theta}-p\right)$$

(with $\frac{1}{\tau_\theta}==\frac{1}{\tau_s}+\frac{1}{\tau_\varepsilon}$)

Informed observe $\widetilde{s}$ so that $\mathbb{E}\left[\widetilde{\theta}/\mathcal{J}_I^0\right]=s$ and var $\left(\widetilde{\theta}/\mathcal{J}_I^0\right)=\sigma_\varepsilon^2$.

So :

$$X_I(p/\mathcal{J}_I^0)=\frac{s-p}{\rho_I\sigma_\varepsilon^2}=t_I\tau_\varepsilon\left(s-p\right)$$

The supply by noise trader is u. So that market clearing gives :

$$(1-\mu)t_U\tau_\theta\left(\overline{\theta}-p\right)+\mu t_I\tau_\varepsilon\left(s-p\right)=u$$

Which gives the price :

$$p=\frac{(1-\mu)t_U\tau_\theta\overline{\theta}+\mu t_I\tau_\varepsilon s-u}{(1-\mu)t_U\tau_\theta+\mu t_I\tau_\varepsilon}$$

The first remark that can be done is that when $\sigma_\varepsilon=0$, that is when the Informed traders are "perfectly" informed, then their demand curve is horizontal $p=s$. The only possible equilibrium is hence $p=s$ perfectly revealing their info.

In the general case, remark that the equilibrium price depends (linearly) on s and u. For instance, when $u=0$, p is larger than $\overline{\theta}$ when s is larger than $\overline{\theta}$, that is when Informed has "a good news about θ". This dependence implies that the price conveys information about s!

Indeed we have at equilibrium:

$$\widetilde{s}=\frac{\left[(1-\mu)t_U\tau_\theta+\mu t_I\tau_\varepsilon\right]\widetilde{p}-\left((1-\mu)\,\tau_\theta t_U\right)\overline{\theta}+\widetilde{u}}{\mu t_I\tau_\varepsilon}$$

So that the best prediction of s varies with p :

$$\mathbb{E}\left(\widetilde{s}/p\right)=\frac{\left[(1-\mu)t_U\tau_\theta+\mu t_I\tau_\varepsilon\right]p-\left((1-\mu)\,\tau_\theta t_U\right)\overline{\theta}}{\mu t_I\tau_\varepsilon}$$

In fact, Uninformed (but sophiticated) traders should have used this information to set their demand (which they have not at this first naïve stage).

But if they modify their demand function accordingly to take into account this information, this will modify the formula of the price equilibrium, function of s and u! this will in turn modify the information inferred by Uninformed...and so on!

The idea of Rational Expectation Equilibrium consists in finding a price formula which is "self fulfilling". If the price formula is $p=f(s,u)$ and if the traders use this information then the equilibrium price will be precisely $f(s,u)$!

Definition 33. A REE is a price function $p=f\,(s,u)$ such that, if traders know this price function, they infer information from price and set their demand accordingly. Doing this it turns out that equilibrium price will be precisely $f(s,u)$. In some sense, this type of equilibrium is the limit of the sequence of inference mentioned above.

4.2.2. Rational Expectation Equilibrium (Grossman and Stiglitz). The idea is to find a linear price formula $p=a+bs-\lambda u$ which is self fulfilling.

Let us be more precise.

Uninformed only observe p :

$$\mathbb{E}\left(\widetilde{\theta}/\mathcal{J}_U\right)=\mathbb{E}\left(\widetilde{\theta}/a+b\widetilde{s}-\lambda\widetilde{u}=p\right)=\mathbb{E}\left(\widetilde{s}+\widetilde{\varepsilon}/b\widetilde{s}-\lambda\widetilde{u}=p-a\right)$$

which gives, using the fact that all variables are normal :

$$\mathbb{E}\left(\widetilde{\theta}/\mathcal{J}_U\right)=\overline{\theta}+\frac{b\sigma_s^2}{b^2\sigma_s^2+\lambda^2\sigma_u^2}\left(p-a-b\overline{\theta}\right)$$

or, equivalently

$$\mathbb{E}\left(\widetilde{\theta}/\mathcal{J}_U\right)=\overline{\theta}+\frac{b\tau_u}{b^2\tau_u+\lambda^2\tau_s}\left(p-a-b\overline{\theta}\right)=\overline{\theta}+\frac{b\tau_u}{k}\left(p-a-b\overline{\theta}\right)$$

In the above formula we have set $k=b^2\tau_u+\lambda^2\tau_s$, (which depends on b and λ).

In the same way :

$$\operatorname{var}\left(\widetilde{\theta}/\mathcal{J}_U\right) = \operatorname{var}\left(\widetilde{s}+\widetilde{\varepsilon}/b\widetilde{s}-\lambda\widetilde{u}=p-a\right) = \sigma_s^2+\sigma_\varepsilon^2-\frac{b^2\sigma_s^4}{b^2\sigma_s^2+\lambda^2\sigma_u^2}$$

$$= \sigma_\varepsilon^2+\frac{\lambda^2\sigma_u^2\sigma_s^2}{(b^2\sigma_s^2+\lambda^2\sigma_u^2)} = \sigma_\varepsilon^2+\frac{\lambda^2}{k}$$

The Uninformed demand is hence :

$$X_U(p/\mathcal{J}_U) = t_U\frac{1}{\sigma_\varepsilon^2+\frac{\lambda^2}{k}}\left[\overline{\theta}+\frac{b\tau_u}{k}\left(p-a-b\overline{\theta}\right)-p\right]$$

The Informed has the same demand function :

$$X_I(p/\mathcal{J}_I) = t_I\tau_\varepsilon\left(s-p\right)$$

Market clearing gives :

$$\mu t_I\tau_\varepsilon\left(s-p\right)+(1-\mu)t_U\frac{1}{\sigma_\varepsilon^2+\frac{\lambda^2}{k}}\left[\overline{\theta}+\frac{b\tau_u}{k}\left(p-a-b\overline{\theta}\right)-p\right] = u$$

Identifying with $p=a+bs-\lambda u$ gives for the coefficients of u and s , and for the constant a :

$$\mu t_I\tau_\varepsilon\lambda+(1-\mu)t_U\frac{1}{\sigma_\varepsilon^2+\frac{\lambda^2}{k}}\left[\frac{-b\lambda\tau_u}{k}+\lambda\right] = 1$$

$$-\mu t_I\tau_\varepsilon b+(1-\mu)t_U\frac{1}{\sigma_\varepsilon^2+\frac{\lambda^2}{k}}\left[\frac{b\tau_u}{k}b-b\right] = -\mu t_I\tau_\varepsilon$$

$$\left(-\mu t_I\tau_\varepsilon-(1-\mu)t_U\frac{1}{\sigma_\varepsilon^2+\frac{\lambda^2}{k}}\right)a+(1-\mu)t_U\frac{\overline{\theta}\tau_s}{\sigma_\varepsilon^2+\frac{\lambda^2}{k}}\left(\frac{\lambda^2}{k}\right) = 0$$

That gives :

$$\lambda\mu t_I\tau_\varepsilon = b \qquad (4.2.1)$$

Such that :

$$p = a+\lambda\left(\mu t_I\tau_\varepsilon s-u\right)$$

reinjecting 4.2.1 in $k=b^2\tau_u+\lambda^2\tau_s$ gives :

$$k = \lambda^2 \left((\mu t_I \tau_\varepsilon)^2 \tau_u + \tau_s \right)$$

We have hence :

$$\text{var}\left(\widetilde{\theta}/J_U\right) = \sigma_\varepsilon^2 + \frac{1}{(\mu t_I \tau_\varepsilon)^2 \tau_u + \tau_s} = \frac{1}{\tau}$$

which allows to find equations giving λ and a :

$$\mu t_I \tau_\varepsilon \lambda + (1-\mu) t_U \tau \left[-\frac{\mu t_I \tau_\varepsilon \tau_u}{\frac{1}{\tau} - \frac{1}{\tau_\varepsilon}} + \lambda \right] = 1$$

$$(-\mu t_I \tau_\varepsilon - (1-\mu) t_U \tau)\, a + (1-\mu) t_U \tau \overline{\theta} \left(\frac{\tau_s}{\frac{1}{\tau} - \frac{1}{\tau_\varepsilon}} \right) = 0$$

It is interesting that $w = s - \frac{1}{\mu t_I \tau_\varepsilon} u$ is the “noisy” information conveyed by p on s. Indeed :
$\mathbb{E}\,(w/s) = s$ and $\text{var}\,(w/s) = \frac{1}{\mu t_I \tau_\varepsilon \tau_u}$

- Informed perfectly informed

Obviously, as in the naïve equilibrium, when τ_ε is infinite (Informed are perfectly informed) then the price gives perfect information on s (and $p = s$).

- No noisy traders

More interestingly, if there are no “noisy” traders $\sigma_u = 0$, then the price gives also perfect information on s. then $\text{var}\left(\widetilde{\theta}/J_U\right) = \sigma_\varepsilon^2 = \text{var}\left(\widetilde{\theta}/J_I\right)$

This gives also $p = s$! Indeed market clearing gives :

$$\mu t_I \tau_\varepsilon (s - p) + (1-\mu)\, t_U \tau_\varepsilon \left[\overline{\theta} + \frac{\left(p - a - b\overline{\theta}\right)}{b} - p \right] = 0$$

which implies $b = 1$ and $a = 0$ as soon as $\mu > 0$.

- No insider

What happens when $\mu = 0$ (no insider)? In that casewe have :

$$\lambda = \frac{\sigma_\theta^2}{t_U}$$

$$a = \overline{\theta}$$

Which means :

$$p = \overline{\theta} - \frac{\sigma_\theta^2}{t_U} u$$

To sum up :

PROPOSITION 34. *In the Rational Expectation Equilibrium, if $\sigma_u = 0$ (no noisy traders), and $\mu > 0$ then the equilibrium price is $p = s$. That means that the market is strongly efficient : the price reflects all public and private information. When $\mu = 0$ the equilibrium price is $\overline{\theta} - \frac{\sigma_\theta^2}{t_U} u$, which gives obviously $p = \overline{\theta}$ when there are no noisy traders.*

REMARK 35. The Grossman Stiglitz paradox. Suppose that prior to trading people decide weather to acquire or not the signal s at a fix cost k, and that there are no noisy traders. If $\mu = 0$ and if k is not too large, it is interesting to buy information : it can be easily shown indeed that the expected utility achieved when informed is larger than the one when everybody is non informed. But as soon as $\mu > 0$, the price becomes fully informative and it is not worth while to buy information since this information becomes free through price!

4.3. Bid ask spread (Glosten and Milgrom)‘

Asymmetry of information can be the source of some market characteristics.

Consider the following model. There is one risky asset whose liquidation value at date 1 is either $\underline{\theta}$ with probability π or $\overline{\theta}$ with probability 1-π. One unit is traded at time zero. There are three types of traders :

- informed traders : they know in advance the value θ at time 1 : they buy if the price is lower than θ and sell if the price is larger.

- liquidity traders: at any price they sell with probability $\frac{1}{2}$ and buy with probability $\frac{1}{2}$.

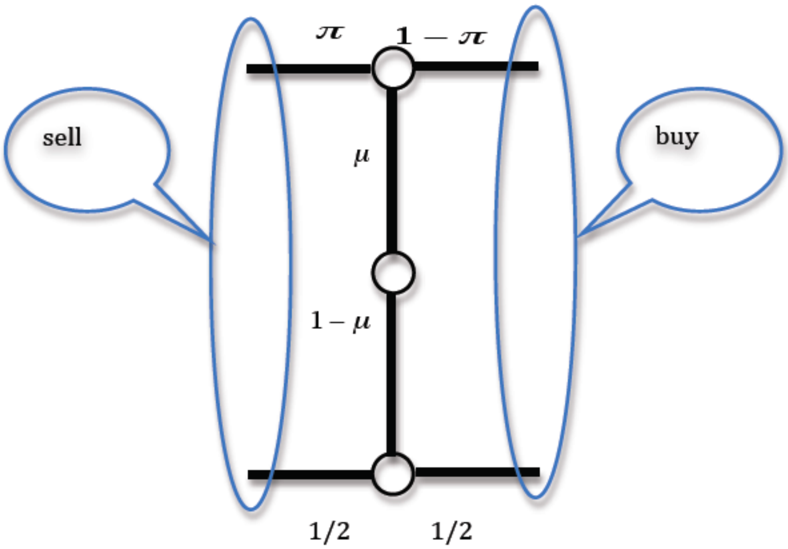
- market makers : they post a bid b (max price to buy) and an ask a (min price to sell) $b \leq a$.

The proportion of informed traders is μ.

The market maker does not know if he faces an informed or a liquidity trader.

Assume $\underline{\theta} \leq b \leq a \leq \overline{\theta}$

We have the following “tree” :



We read the diagram as the following : with probability μ the market maker faces an informed. In this case he will buy if $\theta = \overline{\theta}$ (with probability $1-\pi$) and sell if $\theta = \underline{\theta}$ (with porobability π). With probability $1-\mu$ this is an uninformed who sells or buys with probability $1/2$.

Assume first that $a = b$

With prob $1-\mu$ the market maker faces a non informed trader. The expected profit of the market maker will be : $\frac{1}{2}(a - \mathbb{E}[\theta]) + \frac{1}{2}(\mathbb{E}[\theta] - a) = 0$

With prob μ it is an informed : he buys if $\theta = \overline{\theta}$ and sells if $\theta = \underline{\theta}$

That is with probability $1-\pi$ the profit is $a - \overline{\theta}$ and with probability π the profit is $\underline{\theta} - b$ which are both negative values!

This comes from the asymmetry of information : inferior information of market makers implies negative profit if ask and bid are equal. A bid-ask spread allows to restore non negative

profit.

Indeed assume $\underline{\theta} \leq b < a \leq \overline{\theta}$. It is usefull to set the random variable D "demand" of the trader : this variable has two states $D =$sell and $D =$buy. In the graph above there are four terminal nodes : 2 when $D =$sell and 2 when $D =$buy.

The probabilities of each node is easy to compute.

D	Sell	Buy	total
Informed	$\mu\pi$	$\mu(1-\pi)$	μ
Liquidity	$\frac{1}{2}(1-\mu)$	$\frac{1}{2}(1-\mu)$	$1-\mu$
Total	$\mu\pi+\frac{1}{2}(1-\mu)$	$\mu(1-\pi)+\frac{1}{2}(1-\mu)$	1

The market maker profits are in each case :

Π	Sell	Buy
Informed	$\overline{\theta}-a$	$b-\underline{\theta}$
Liquidity	$\mathbb{E}[\theta]-a$	$b-E[\theta]$

To determine the values of a and b, Glosten and Milgrom assume that the "average" (expected) profit made when he sells (resp buys) is zero :

$$\mathbb{E}[\Pi/D]=0$$

We have $\Pr(\text{Informed}/D=\text{sell})=\frac{\mu\pi}{\mu\pi+\frac{1}{2}(1-\mu)}$ and $\Pr(\text{Liquidity}/D=\text{sell})=\frac{\frac{1}{2}(1-\mu)}{\mu\pi+\frac{1}{2}(1-\mu)}$

So that $\frac{\mu\pi}{\mu\pi+\frac{1}{2}(1-\mu)}\left(\overline{\theta}-a\right)+\frac{\frac{1}{2}(1-\mu)}{\mu\pi+\frac{1}{2}(1-\mu)}(\mathbb{E}[\theta]-a)=0$ which implies :

$$a=\frac{\mu\pi\overline{\theta}+\frac{1}{2}(1-\mu)\mathbb{E}[\theta]}{\mu\pi+\frac{1}{2}(1-\mu)}$$

In the same way :

$\frac{\mu(1-\pi)}{\mu(1-\pi)+\frac{1}{2}(1-\mu)}(b-\underline{\theta})+\frac{\frac{1}{2}(1-\mu)}{\mu(1-\pi)+\frac{1}{2}(1-\mu)}(b-E[\theta])=0$

$$b=\frac{\mu(1-\pi)\underline{\theta}+\frac{1}{2}(1-\mu)\mathbb{E}[\theta]}{\mu(1-\pi)+\frac{1}{2}(1-\mu)}$$

When μ is large, a is close to $\overline{\theta}$ and b is close to $\underline{\theta}$. Conversely when μ is small, a and b are close to $\mathbb{E}(\theta)$.

4.4. The capital asset pricing model

4.4.1. The principle of diversification. The principle of diversification is a central principle in finance and insurance.

The purpose of this first section is to give a meaning to the phrase "do not put all your eggs in one basket" , which is the popular expression of the principle of diversification.

The idea of diversification principle is actually quite simple, and makes a clear statement: (unless they are perfectly correlated) half the sum of two identically distributed random variables, having a finite expectation, is less risky than each of them. for example, if we imagine two baskets each with the same probability p to fall (and thus cause the loss of eggs) , put an egg in each basket is less risky than putting them both in one .

Indeed in the first case it will be possible to eat 0 eggs with probability p^2 , 2 eggs with probability $(1-p)^2\}$, And 1 egg with probability $2p(1-p)$. In the second 0 with probability p and 2 with probability $1-p$. The probabilities of the extreme, 0 and 2 , decreased : p to p^2 a decrease of $p(1-p)$) and $(1-p)$ to $(1-p)^2$, while the medium event, 1 egg , has its probability increased of exactly $2p(1-p)$.

If $\widetilde{X}$ is the random variable giving 1 if the first basket remains intact and 0 if it falls, $\widetilde{Y}$ defined in the same manner for the second rack , is a $2\widetilde{X}$ and 2 $\widetilde{Y}$ are riskier than $\widetilde{X}+\widetilde{Y}$.

This notion of risk reduction is associated to a "concentration" of the probability distribution function. It can be shown that if $\widetilde{A}$ and $\widetilde{B}$ have the same finite expectation, $\widetilde{A}$ is more risky than $\widetilde{B}$ if and only if there exists a random variable $\widetilde{\varepsilon}$ such that $\widetilde{A}=\widetilde{B}+\widetilde{\varepsilon}$ with $\mathbb{E}\left(\widetilde{\varepsilon}/\widetilde{B}\right)=0$. $\widetilde{A}$ is a noisy transformation of $\widetilde{B}$. Obviuously, if the variables have finite variances this implies that $\operatorname{var}\left(\widetilde{A}\right)\geq\operatorname{var}\left(\widetilde{B}\right)$

In the case of n independent and identically distributed random variables we can state the following general result :

Proposition 36. *if $\widetilde{x}_i$ are n independent real random variables and identically distributed such that $|\mathbb{E}\left(\widetilde{x}_i\right)|<+\infty$, then $\forall\alpha_i$, N positive real numbers of sum 1 ($\sum\alpha_i=1$) $\widetilde{x}_{\frac{1}{n}}=\frac{\sum\widetilde{x}_i}{n}$ is less risky than $\widetilde{x}_\alpha=\sum\alpha_i\widetilde{x}_i$. (and in particular than each of the $\widetilde{x}_i$) : that means that the probability distribution of $\widetilde{x}_{\frac{1}{n}}$ is more concentrated than that of $\widetilde{x}_\alpha$ or alternatively that, for all α, there exists $\widetilde{\varepsilon}$ such that $\widetilde{x}_\alpha=\widetilde{x}_{\frac{1}{n}}+\widetilde{\varepsilon}$ with $\mathbb{E}\left(\widetilde{\varepsilon}/\widetilde{x}_{\frac{1}{n}}\right)=0$.*

In particular if the variables have a finite variance σ^2 we have :

$$\operatorname{var}\left(\widetilde{x}_\alpha\right)=\left(\sum\alpha_i^2\right)\sigma^2$$

Which is minimum for $\alpha_i=\frac{1}{n}$

Obviously the situation is slightly more complex when the random variables are correlated. First, consider two random variables $\widetilde{x}_1$ and $\widetilde{x}_2$ with the same variance σ^2 but not necessarily independent . A study of variance allows to get an idea of the risk of a convex combination of the two variables.

Consider $t(\alpha)$:

$$t(\alpha)=\frac{\operatorname{var}\left(\alpha\widetilde{x}_1+(1-\alpha)\widetilde{x}_2\right)}{\sigma^2}$$

$$=\alpha^2+(1-\alpha)^2+2\alpha(1-\alpha)\frac{\operatorname{cov}(\widetilde{x}_1,\widetilde{x}_2)}{\sigma^2}$$

$t(\alpha)$ is less than 1, because $\operatorname{cov}(\widetilde{x}_1,\widetilde{x}_2)<\sigma^2$. it is minimal for $\alpha=1/2$.

But now if , $\widetilde{x}_1$ and $\widetilde{x}_2$ have not the same variance, with for example $\operatorname{var}(\widetilde{x}_1)\leq\operatorname{var}(\widetilde{x}_2)$. One computes :

$$\Sigma^2(\alpha) = \text{var}\left(\alpha\widetilde{x}_1 + (1-\alpha)\widetilde{x}_2\right)$$

$$= \alpha^2\text{var}(\widetilde{x}_1) + (1-\alpha)^2\text{var}\left(\widetilde{x}_2\right) + 2\alpha(1-\alpha)\ \text{cov}(\widetilde{x}_1, \widetilde{x}_2)$$

$$= \alpha^2\left(\text{var}(\widetilde{x}_1) + \text{var}\left(\widetilde{x}_2\right) - 2\text{cov}(\widetilde{x}_1, \widetilde{x}_2)\right) + 2\alpha\left(\text{cov}(\widetilde{x}_1, \widetilde{x}_2) - \text{var}\left(\widetilde{x}_2\right)\right) + \text{var}\left(\widetilde{x}_2\right)$$

As this must be always positive, the discriminant must be negative :

$$|\text{cov}(\widetilde{x}_1, \widetilde{x}_2)| \leq \sqrt{\text{var}\left(\widetilde{x}_1\right)\text{var}\left(\widetilde{x}_2\right)} \leq \text{var}\left(\widetilde{x}_2\right)$$

This means that the correlation coefficient $\frac{\text{cov}(\widetilde{x}_1,\widetilde{x}_2)}{\sqrt{\text{var}(\widetilde{x}_1)\text{var}(\widetilde{x}_2)}}$ lies between -1 and 1!

We have :

$$\frac{d\Sigma^2}{d\alpha} = \alpha\left(var(\widetilde{x}_1) - \text{cov}(\widetilde{x}_1, \widetilde{x}_2)\right) - (1-\alpha)\left(var(\widetilde{x}_2) - \text{cov}(\widetilde{x}_1, \widetilde{x}_2)\right)$$

The minimum is obtained for :

$$\frac{\alpha^*}{1-\alpha^*} = \frac{var(\widetilde{x}_2) - \text{cov}(\widetilde{x}_1, \widetilde{x}_2)}{var(\widetilde{x}_1) - \text{cov}(\widetilde{x}_1, \widetilde{x}_2)}$$

That is

$$\alpha^* = \frac{var(\widetilde{x}_2) - \text{cov}(\widetilde{x}_1, \widetilde{x}_2)}{var(\widetilde{x}_1) + var(\widetilde{x}_2) - 2\text{cov}(\widetilde{x}_1, \widetilde{x}_2)}$$

α^* belongs to $[0, 1]$ if :

$$\text{var}\left(\widetilde{x}_1\right) - \ \text{cov}(\widetilde{x}_1, \widetilde{x}_2) \geq 0$$

That we write :

$$\frac{\text{cov}(\widetilde{x}_1, \widetilde{x}_2)}{\text{var}\left(\widetilde{x}_1\right)} \leq 1$$

$\frac{\text{cov}(\widetilde{x}_1,\widetilde{x}_2)}{\text{var}(\widetilde{x}_1)}$ is the "beta" coefficient $\beta(\widetilde{x}_2/\widetilde{x}_1)$ of 2 with respect to 1 : even if variance of 2 is larger than that of 1, a combination of the two assets allows to decrease the risk below that of 1 if $\beta(\widetilde{x}_2/\widetilde{x}_1) \leq 1$. If $\tau = \frac{\text{cov}(\widetilde{x}_1,\widetilde{x}_2)}{\sqrt{\text{var}(\widetilde{x}_1)\text{var}(\widetilde{x}_2)}}$ is the correlation coefficient, this means that the risk can be diminihed below the lowest risk if $\tau\sqrt{var\left(\widetilde{x}_2\right)} \leq \sqrt{var\left(\widetilde{x}_1\right)}$ (which is always true if $\tau \leq 0$).

However, if it is not the case, the minimal risk is obtained for $\alpha \notin [0, 1]$.

4.4.2. Portfolio choice. In this section we propose to analyze the problem of portfolio choice in general . An investor has 1 euro, how should it be allocated among the various assets available? Obviously the answer depends on his attitude to risk. We will assume here that our investor uses the mean-variance criterion, i.e. he evaluates $\mathbb{E}(\widetilde{v}) - \frac{1}{2}\theta\text{var}(\widetilde{v})$ to compare the random variables. This is the case, in particular, when all variables are gaussian and the decision maker has a concave utility function.

Consider K financial assets ,$k = 1, ..., K$. Income from asset k is a real random variable $\widetilde{a}_k$: the income (cash) that provides the risky asset in the future. This random variable is assumed to be known through statistical studies. We note p_k the market price of asset k . It is quite convenient to define the return on assets k as the random variable that measures the income for one euro : $\widetilde{R}_k = \frac{\widetilde{a}_k}{p_k}$. There is also a risk-free asset , the asset 0 , which givesR_0 (nonrandom) euros per euro invested . A risky portfolio is a vector θ, each component θ_k measuring the amount of asset k held.

The random income of a portfolio is $\sum \widetilde{a}_k\theta_k$ and its cost ${}^t\theta p$.

One can write the income as a function of yields :

$$\widetilde{v} = \frac{\widetilde{a}_k}{p_k} p_k\theta_k = \widetilde{R}_k x_k = {}^t x\widetilde{R}$$

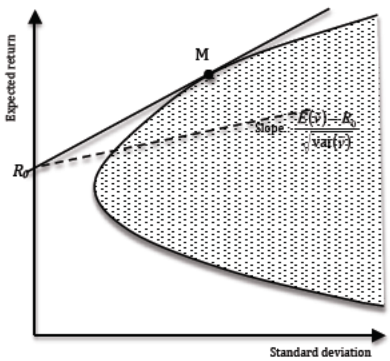
Where$x_k = p_k\theta_k$ is the expenditure to purchase the asset k.

Suppose our investor has a euro to be shared among different assets. He must choose to spread this euro between risk-free asset (x_{0}) and risky assets$x = (x_1, ...x_K)$ with $\mathbf{1}$ is the vector with$K$ components all equal to 1 :$x_0 + {}^t x\mathbf{1} = 1$. This strategy gives him for one euro , an income equal to $x_0 R_0 +^t x\widetilde{R}. = R_0 + {}^t x(\widetilde{R} - R_0\mathbf{1})$.

We se t$\widetilde{\rho} = \widetilde{R} - R_0\mathbf{1}$, the vector of excess returns over the risk-free asset . Income from invested euro is equal to $R_0 + {}^t x\widetilde{\rho}$.

The problem for the investor is to choose the optimal x.

Before doing computations, use a graphical reasonning. The following graph represents each portfolio by a point whose coordinates are the standard deviation (that is the root square of the variance $\sqrt{\text{var}(\widetilde{v})}$) of its return on the horizontal axis and the expectation $\mathbb{E}(\widetilde{v})$ on the vertical.



The shaded area is the set of all the possible portfolios containing only risky assets. Consider then a portfolio in that area and combinate it with the risk free asset. Say for instance that you put $1 - x_0$ euros on this portfolio and x_0 on the risk free asset. The expectation will be $\mathbb{E}\left(\widetilde{w}\right) = x_0 R_0 + (1-x_0)\mathbb{E}\left(\widetilde{v}\right)$ and the standard deviation $\sqrt{\text{var}\left(\widetilde{w}\right)} = (1-x_0)\sqrt{\text{var}\left(\widetilde{v}\right)}$. The expected excess return is $\mathbb{E}\left(\widetilde{w}\right) = (1-x_0)\left(\mathbb{E}\left(\widetilde{v}\right) - R_0\right)$ so that this strategy is such that $\frac{\mathbb{E}(\widetilde{w})-R_0}{\sqrt{\text{var}(\widetilde{w})}} = \frac{\mathbb{E}(\widetilde{v})-R_0}{\sqrt{\text{var}(\widetilde{v})}}$. The point obtained is on the line between the point $(0, R_0)$ and the point $\left(\sqrt{\text{var}\left(\widetilde{v}\right)}, \mathbb{E}\left(\widetilde{v}\right)\right)$ whose slope is exactly $\frac{\mathbb{E}(\widetilde{v})-R_0}{\sqrt{\text{var}(\widetilde{v})}}$ which is called “the Sharpe ratio” of the portfolio. Hence, all the possible combinations are obtained by drawing lines between the shaded area and the point $(0, R_0)$.

For portfolio with a given standard deviation (a given risk) we seek those with the maximum expected return. To do this consider the point M which has the maximal Sharpe ratio in the shaded area. combining this portfolio with the risk free asset give portfolios on the bold line. Obviouly, the region above this line cannot be reached : there are no feasible combination that gives these expectation and standard deviation. Conversely, the points below are all feasible.

The bold line is hence the “efficiency frontier” of the market : efficient points (those with maximal expected return at agiven risk) are on this line. M is the market portfolio : all the investors share their investment between the risk free asset and this particular portfolio.

This result can be derived more formally.

We have :

$$\left\{ \begin{array}{c} E(\widetilde{w}) = R_0 + {}^t x E(\widetilde{\rho}) \\ \text{var}(\widetilde{w}) = E\left[(\widetilde{v} - E(\widetilde{v}))^2\right] \end{array} \right.$$

$$\left\{ \begin{array}{l} \text{var}(\widetilde{w}) = E\left[({}^t x\widetilde{R} - {}^t xE(\widetilde{R}))^2\right] \\ = E\left[{}^t x(\widetilde{R} - E(\widetilde{R}))^t(\widetilde{R} - E(\widetilde{R}))x\right] \\ = {}^t xE\left[(\widetilde{R} - E(\widetilde{R}))^t(\widetilde{R} - E(\widetilde{R}))\right] x \end{array} \right.$$

set

$$\Omega = E\left[(\widetilde{R} - E(\widetilde{R}))^t(\widetilde{R} - E(\widetilde{R}))\right]$$

Ω is called the matrix (symmetric) of variance-covariance of assets. The ij element equals $\sigma_{ij} = E\left[(\widetilde{R}_i - E(\widetilde{R}_i))(\widetilde{R}_j - E(\widetilde{R}_j))\right]$ σ_{ij} is the covariance between assets i and j . The formula above shows that this symmetric matrix is positive (the associated quadratic form is positive : $x^t\Omega x$ is a variance , which is positive) .

PROPOSITION 37. *To sum up, the strategy (x_0, x) gives $R_0 +^t xE(\widetilde{\rho})$ on average with a variance equal to ${}^t x\Omega x$* .

In the following if $\widetilde{v}$ is a random variable , we note v its expectation. Here $\rho_i = E(\widetilde{\rho}_i), \quad R_i = E(\widetilde{R}_i)$.

How to choose between all possible strategies ? Clearly two strategies give the same expectation , any risk-averse investor prefers the strategy of minimum variance. Therefore fix our investor expected return at m and seek the vectors x of $\mathbb{R}^K$ that minimizes the variance and give m as expected return . Consider the optimization problem (P) :

$$(P)\begin{cases} \min_x({}^t x\Omega x) \\ {}^t x\rho + R_0 = m \end{cases}$$

Define a new scalar product :

DEFINITION 38. $< x, y >= {}^t x\Omega y$ is a scalar product (quadratic positive definite form) note $\| x \|$, the associated norm.

The problem becomes :

$$(P)\begin{cases} \min_x \| x \| \\ < \Omega^{-1}\rho, x >= m - R_0 \end{cases}$$

This amounts to find the point of the affine hyperplane ($< \Omega^{-1}\rho, x >= m - R_0$), that is the closest from 0. This point is the orthogonal projection x^* of 0 on this hyperplane . It is defined by two equations with unknown x^* and λ :

$$\begin{cases} x^* = \lambda\Omega^{-1}\rho \\ < \Omega^{-1}\rho, x^* >= m - R_0 \end{cases}$$

The first one says that x^* is colinear with the orthogonal vector $\Omega^{-1}\rho$ of the hyperplane, The second says that tis projection belongs to this affine hyperplane.

- An important note should be made x^* is a vector which is proportional to the vector $\Omega^{-1}\rho$ which does not depend on m . In other words, regardless of the expected return required , the structure of the risky portfolio is identical. Structure refers to the relative proportion of different risky assets .
- Of course we can easily solve the above system :

$$x^* = \frac{(m - R_0)}{{}^t\rho\Omega^{-1}\rho}\Omega^{-1}\rho$$

- How our investor chooses the level he m ? Obviously this is determined by his tradeoff between mean and variance , maximize wrt to m :

$$E(\widetilde{w}) - \frac{1}{2}\theta var(\widetilde{w}) = m - \frac{1}{2}\theta(m - R_0)^2\frac{1}{{}^t\rho\Omega^{-1}\rho}$$

DEFINITION 39. One calls market portfolio the portfolio $x^m = \frac{\Omega^{-1}\rho}{{}^t\mathbf{1}\Omega^{-1}\rho} = \mu\Omega^{-1}\rho$, that portfolio contains only risky assets in the relative proportions defined by the solutions of problems (P). The yield of this portfolio is : $\widetilde{R}_m = R_0 + {}^t x^m\widetilde{\rho}$, The variance is :

$$var(\widetilde{R}_m) = {}^t x^m\Omega x^m$$

Moreover:

$$\begin{aligned}(\Omega x^m)_i &= \sum_j \sigma_{ij}x_j^m \\ &= cov(\widetilde{R}_i, \sum \widetilde{R}_j x_j^m) \\ &= cov(\widetilde{R}_i, \widetilde{R}_m)\end{aligned}$$

This portfolio is called the market portfolio because, under the investors mean-variance assumption , the above shows that all individuals demand a portfolio whose risky component is proportional to this portfolio. It follows that the total demand of all the portfolios held the same structure (in their risky part) . Of course, a very risk-averse individual will ask a relative little amount of this risky portfolio and focus his investment on the risk-free asset . Instead, a less risk-averse individual will choose a x_0 smaller. It is as if each investor was buying a piece of the total market capitalization , piece more or less according to risk aversion !

4.4.3. Capital asset pricing model formula. The above leads to find one of the most famous of finance formulas.

One has :

$$\begin{aligned}\Omega x^m &= \mu\rho \\ var(\widetilde{R}_m) &= {}^t x^m\Omega x^m = \mu^t x^m\rho = \mu(R_m - R_0)\end{aligned}$$

This implies :

$$\frac{\Omega x^m}{var(\widetilde{R}_m)}(R_m - R_0) = \rho$$

Coordinate by coordinate:

$$\frac{cov(\widetilde{R}_i, \widetilde{R}_m)}{var(\widetilde{R}_m)}(R_m - R_0) = (R_i - R_0)$$

PROPOSITION 40. *For any asset i its average outperformance over the risk-free asset* $R_i - R_0$ *is proportional to the outperformance of the market portfolio*$R_m - R_0$. *The proportionality factor is the coefficient* β_i *, relative to the market portfolio .*

$$\begin{aligned} R_i - R_0 &= \beta_i(R_m - R_0) \\ \beta_i &= \frac{cov(\widetilde{R}_i, \widetilde{R}_m)}{var(\widetilde{R}_m)} \end{aligned}$$