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An Equilibrium Model of Catastrophe Insurance Futures and Spreads

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Abstract

This article presents a valuation model of futures contracts and derivatives on such contracts, when the underlying delivery value is an insurance index, which follows a stochastic process containing jumps of random claim sizes at random time points of accident occurrence. Applications are made on insurance futures and spreads, a relatively new class of instruments for risk management launched by the Chicago Board of Trade in 1993, anticipated to start in Europe and perhaps also in other parts of the world in the future. The article treats the problem of pricing catastrophe risk, which is priced in the model and not treated as unsystematic risk. Several closed pricing formulas are derived, both for futures contracts and for futures derivatives, such as caps, call options, and spreads. The framework is that of partial equilibrium theory under uncertainty.

Key words: insurance futures, futures derivatives, claims processes, reinsurance

1. Introduction

This article presents a valuation model of futures contracts and derivatives on such contracts when the underlying "delivery" value is an insurance index that follows a stochastic process containing jumps of random claim sizes at random points of accident occurrence. The article deals with the delicate problem of pricing *catastrophe risk*, which is priced in this model and not treated as unsystematic risk. In the representation of the loss ratio index, we follow insurance tradition by using a standard actuarial approach. The presented model combines economic theory with actuarial practice and theory.

The newly founded market for insurance derivatives is the motivation behind this article. This market was established in December 1992 by the Chicago Board of Trade (CBOT) and offers insurers an alternative to reinsurance as a hedging device for underwriting risks. The terminal cash flow is related to the aggregate claims incurred during a calendar quarter or, more precisely, to moves in a loss ratio based on figures for claims and premiums compiled by a statistical agent (Insurance Services Office (ISO)). The settlement price for each futures contract increases by $250 for each percentage point upward movement in the ratio.

In September 1995 the CBOT introduced a new class of catastrophe insurance options based on new insurance indices provided by Property Claim Services, a division of American Insurance Services Group, Inc. The latter options are called *PCS*[TM] Catastrophe Insurance Options, or PCS options for short. The underlying index represents the development of specified catastrophe damages and is calculated and published daily by PCS.

Another interesting innovation is a set of new agricultural insurance contracts, approved by the CBOT on October 18, 1994, that started trading the same year. These contracts—known as *area yield options*—provide a means for hedging against shortfall in the harvest of particular crops. An advantage of the crop yield contracts is that there is already an OTC derivatives market in this area. More ambitious OTC deals are on the drawing board. For example, it would be possible to devise instruments that would effectively swap hurricane risk for earthquake risk. Another possibility is "act of God" bonds with coupons that decline as the number of catastrophe insurance claims rises. Insurers would be natural issuers of such products. With all these new instruments, in addition to the old ones, markets may appear to be getting closer and closer to a complete Arrow-Debreu market.

The present article presents a situation where the underlying stochastic dynamics is assumed to allow for unpredictable jumps at random time points. In particular, we have in mind claims caused by accidents in an insurance framework, as the loss ratio indices in the CBOT exchange,[1] and we intend to model an index of such claims by a random, marked point process. An arbitrage pricing model based on this assumption usually contains many equivalent martingale measures, so this approach does not lead to a unique pricing rule. Although some progress has been made in this direction (see, e.g., Föllmer and Sondermann [1986]; Schweizer [1991], we choose to stay within the framework of partial equilibrium theory and derive prices of forward contracts and relevant derivatives within this setting. There is, however, an arbitrage type approach given by Cummins and Geman [1995] for catastrophe insurance futures. They use the time integral of geometric Brownian motion plus jumps terms as a model for the accumulated claims index to price futures and call spreads and use an Asian options approach.

Unlike reinsurance, hedging through futures has the advantage of reversibility since any position may be closed before the maturity of the contract. In principle, a traditional reinsurance contract may be reversed; however, in practice reversing a reinsurance transaction exposes the insurer to relatively high transaction costs, presumably to protect the reinsurer against adverse selection. Furthermore, there should be the advantages of liquidity associated with ordinary futures markets. Consequently, this new market is likely to improve welfare.

The article is organized as follows. In Section 2 the catastrophe insurance market is briefly discussed. In Section 3 the economic model is presented, as well as the equilibrium pricing results to be utilized. In Section 4 we present the first of the main results, a very simple futures pricing formula, allowing properties of the model to be derived, and discussed. In Section 5 pricing formulas for derivatives on the futures are derived, where we analyze in detail a futures cap, a futures call option, and a futures spread. In Section 6 we offer some concluding remarks.

2. The catastrophe insurance market

In this section we intend to motivate the article by briefly discussing futures contracts on insurance indexes and, in particular, the contracts offered by the CBOT. We do not intend to give a very detailed description of this market, but we would like to present at least the skeleton necessary to understand the principles. The reader could consult Cummins and

Geman [1995] for a more detailed account and also the material published by the CBOT, mentioned in the references.

2.1. The CBOT market

Before the insurance risk can be securitized, it must be standardized. In the case of the CBOT's catastrophe insurance contracts, this meant devising an index on which to base derivatives. Unlike the equity, bond, or commodity markets, the insurance market has no obvious, continuously updated underlying cash price. The solution chosen by the Board of Trade is the loss ratio index, which was calculated by the Insurance Services Office for the CAT products and is provided by the Property Claim Services for the PCS options. In the former case, which we now discuss briefly, data were used from at least ten designated reporting companies. The loss ratio is the dollar value of reported losses incurred in a given quarter (the loss quarter) and reported by the end of the following quarter (the run-off quarter) divided by one-fourth of the dollar value of the premiums collected in the previous year. The contract value is $25,000 times the loss ratio.

The initial offerings were limited to catastrophic property-insurance losses, and the two instruments initially introduced covered national and Eastern property catastrophes. Contracts covering Midwestern and Western catastrophes were added during 1993. Regional futures are viewed as potentially important because of the differing catastrophe exposure in the different regions. Property insurance is an important branch of insurance and has the advantage that losses settle relatively quickly, unlike some other types of insurance such as commercial liability or workers' compensation. The index created consists of losses reported each quarter to the ISO, and this forms the basis for futures trading. Approximately 100 companies report property loss data to the ISO. The settlement values for insurance futures are based on losses incurred by a pool of a least ten of these companies selected by the ISO on the basis of size, diversity of business, and quality of reported data. In addition to announcing the list of companies included in the pool for any given futures contract, the CBOT also announces the premium volume for the companies participating in the pool prior to the start of the trading period for each catastrophe contract. Thus, the premiums in the pool is a known constant throughout the trading period, and price changes are attributable solely to changes in the market's attitude toward risk and expectations of loss liabilities at each time t, given the available information at that time, as is the case of ordinary futures, where the underlying commodity price fluctuates randomly, and so do the associated futures prices.

Catastrophe insurance futures trade on a quarterly cycle, with contract months March, June, September, and December. A contract for any given calendar quarter is based on losses occurring in the listed quarter that are reported to the participating companies by the end on the following quarter. The six-month period following the start of the calendar quarter is known as the "reporting period." The three additional reporting months following the close of the event quarter are to allow for loss settlement lags. Although not all losses will be reported by the end of the two-quarter reporting period, reported pool losses should represent a high proportion of eventual paid losses, particularly in view of the fact that companies are allowed to report incurred (paid plus estimated unpaid) losses. Unlike most reinsurance arrangements, insurance futures do not focus on a particular type of policy but rather on particular types of losses. Losses included in the pool consist of all property losses incurred by the reporting companies arising from perils of windstrom, hail, earthquake, riot, and flood. Reported losses can arise from eight different lines of insurance including homeowners, commercial multiple peril, earthquake, and automobile physical damage. Even though the contracts are called catastrophe futures, in fact all losses for the specified perils and line of business are included in the loss pool. However, the losses in the pool are expected to be highly correlated with property catastrophe losses because the included perils were chosen as those most susceptible to catastrophe. The use of a proxy index rather than the true catastrophe losses stems from reporting difficulties and semi-long-tail problems common in most lines of insurance.

The idea for the insurers is to use this market to hedge against unexpected losses in the following quarter. Clearly, if the loss ratio of the pool is not perfectly correlated with that of the insurer, this hedge will not be perfect. The splitting of the index into different regions, with some common pattern of risk exposures within each region, and with the risk inhomogeneity being between the regions, is clearly an advantage toward making the hedge more effective. Moreover, if the realized losses are lower than anticipated, the insurer incurs a loss from a possible long position. As an alternative, the insurer can buy a call option, or a spread, on the CBOT futures index, thereby having the ability to profit from losses that are less than originally forecasted. Of course, this requires a cash payment at the beginning of the event quarter, but this seems more in line with traditional insurance, where premiums are normally paid in advance. For insurers with expectations of a moderate increase in the CBOT index relative to the market's expectations based on the ISO reporting companies, it seems reasonable to buy a bull spread on the CBOT futures index, corresponding very much to a "layer" of protection in reinsurance terminology.

The opposite side of the market (sellers of futures) consists primarily of investors and speculators. For insurers with risk profiles different from the sample of ISO reporting companies, there may be an element of speculation in trading in the CBOT futures index as well. As we have noted, a perfect hedge can only be obtained using traditional reinsurance, but since insurers' business is precisely that of risk bearing, they may normally not be interested in a "perfect" hedge,[2] since the best they can hope for then is profits close to the riskless rate, which is not likely to satisfy most stockholders of insurance companies. Partly due to moral hazard and adverse selection, a perfect hedge can be expensive, and sometimes traditional insurance is impossible to obtain. This new futures insurance market may therefore be an excellent innovation in the insurance business, and possibly improve welfare at large through better risk sharing and risk distribution, at least when combined with traditional insurance/reinsurance. We should add, though, that the way the loss ratio is constructed may lead to a moral-hazard problem, since the same companies responsible for the construction of the loss ratio index also trade in this market. We shall ignore possible effects from this in our model.

We will not describe the detailed institutional basics of PCS options or CAT products here, but only comment on certain issues along the way where they may be of importance for the development and understanding of the model.[3] We notice that PCS loss indices represent the absolute USD amount of those catastrophe damages that are estimated and published by PCS for a defined region and a defined time period, whereas CAT products, as explained above, refer to an underlying loss ratio index. In the PCS definition, catastrophe damages are losses of more than 5 million of insured property damages which affect at the same time a significant number of insurance companies and policy holders. The definition of CAT catastrophes depends on specific loss causes, where a minimum loss volume is not necessary.

A likely scenario is that risk-averse insurers are seeking some “reinsurance” protection in the CBOT futures index and thereby are willing to pay a risk premium for this protection. On the other side are the investors requiring compensation for bearing risk, and in market equilibrium prices are presumably determined such that these risk premiums and profits average out in the long run (abstracting here from transaction costs). From this speculation we would expect to get market futures prices that are larger than expected values. In any case, risk premiums must be endogenous in any realistic model of this market, which we accomplish in our model.

3. The economic model

Economists have developed during the last thirty years a canonical model to deal with optimal insurance/risk-sharing and risk prevention. Our aim in this section is to review the assumptions and basic results of this simple model. Later we apply the principles of this theory to a model of a catastrophe futures and derivatives market.

3.1. Introduction

To explain the model to be used, it may be helpful to first consider a reinsurance syndicate consisting of I members, where each member is characterized by a utility function $U^{(i)}$ and net reserves $X^{(i)}(t) = W_t^{(i)} + \int_0^t a^{(i)}(s)\,ds - Z^{(i)}(t)$, where $W_t^{(i)}$ is the consumption by time t of insurer i, $a^{(i)}(t)$ is a premium rate, and where

$$Z^{(i)}(t) = \sum_{k=1}^{N^{(i)}(t)} Y_k^{(i)}, \quad i = 1, 2, \ldots, I. \tag{1}$$

Here $N^{(i)}(t)$ is the number of claims incurred by time t in the portfolio of insurer i, and $Y_k^{(i)}$ is the size of the kth claim (occurring at the random time $\tau_k^{(i)}$).

We now define an index of insurance claims as an aggregate—that is, let

$$X(t) = W_t + \int_0^t a(s)\,ds - Z(t), \tag{2}$$

where W_t represents the aggregate consumption in the market by time t, and $\int_0^t a(s)\,ds =$ premiums earned in the market by time t^4. In general, a_t is a (possibly time-varying) premium rate, which may be a bounded variation, $\mathcal{F}_t$-predictable stochastic process,[5] but later we shall by and large simplify by consider the case $a(t) \equiv a > 0, a$ constant. The length of the calendar quarter (the "event quarter") equals T, and Π, the total premiums for the next quarter, is considered prepaid and known before the event quarter starts. In the case where $a(t)$ is some constant a, $\Pi = aT$. We make the assumption that W_t is independent of Z_t for each t.

The stochastic process $Z(t)$ represents the aggregate claims by time t reported to, say, the ISO pool. Since claims, and in particular catastrophes, cannot be considered as infinitesimal, we represent Z by a process of the type considered in Eq. (1)[6]

$$Z(t) = \sum_{k=1}^{N(t)} Y_k, \tag{3}$$

where $N(t)$ is the number of claims incurred by time t, and where Y_k is the size of the kth claim occurring at the random time τ_k. This interpretation will be used in the rest of the article. We shall assume that $Z(t)$ is a compound Poisson process, implying that $N(t)$ is a Poisson process with intensity λ, and $Y_1, Y_2, \ldots$ are independent (nonnegative) random variables, all independent of the process N. For the claim size distribution (the probability distribution of the Y_k), we shall assume it to be gamma with parameters (n, μ), where n is some given positive integer and where μ is a positive real number. (For more on this model in reinsurance markets, see, e.g., Aase [1992, 1993d]).

These random variables are all defined on the same complete probability space $(\Omega, \mathcal{F}, P)$. The economy has a finite horizon $\mathcal{T} = [0, T]$. The flow of information is given by a natural filtration—that is, the augmented filtration $\{\mathcal{F}_t; t \in \mathcal{T}\}$ of σ-algebras of $\mathcal{F}$ generated by X. Let L be the set of adapted processes Y satisfying the integrability constraint

$$E\left(\int_0^T Y(t)^2\, dt\right) < \infty. \tag{4}$$

The preferences of the agents we assume to be represented by additively separable von Neumann-Morgenstern utility functions of the form

$$U^i\big(X^{(i)}\big) = E\left\{\int_0^T u_i\big(X^{(i)}(t), t\big)\, dt\right\}, \quad i = 1, 2, \ldots, I \tag{5}$$

where $u_i'(x, t) = e^{-\alpha_i x - \rho_i t}$. Here α_i is the coefficient of absolute risk aversion and ρ_i the time impatience rate of agent i, respectively. We then know that the aggregation property holds (see, e.g., Borch [1962], Wilson [1968] or Rubinstein [1974]), and we obtain a representative agent of the form

$$U(X) = E\left\{\int_0^T u(X(t), t)\, dt\right\}, \tag{6}$$

where $u'(x, t) = e^{-\alpha x - \rho t}$, where α is now the intertemporal coefficient of absolute risk aversion in the market, a constant here assumed smaller than the parameter μ from the claim size distribution of the aggregated claims, and where ρ is the subjective time impatience rate in the market as a whole.

The model $X(t) = w + at - Z(t)$ is called the Lundberg, or the Cramer-Lundberg model, where w is some constant and is well-known in insurance. Because of the presence of unpredictable jumps at random time points in our model, linear spanning cannot be expected to hold for any $Z \in L$. By linear spanning we here mean that any such random variable Z can be represented in the form $Z = Z_0 + \int_0^T \theta(s)\, dM(s)$ for $M_t = X_t - pt$ for some constant p, where M is a compensated martingale associated with X, under some equivalent martingale measure Q (see Section 3.3 below), and θ is some predictable strategy satisfying (4).[7] Thus our model is incomplete. For the futures contract we analyze in Section 4 spanning of the above kind would be difficult to utilize in practice, since the index itself cannot be bought.

We may appeal to the *aggregation theorem* of Rubinstein [1974]. It relies on the observation that under certain conditions, even though the market may be incomplete, the equilibrium prices will be determined as if there were an otherwise similar complete (Arrow-Debreu) market. Under our conditions with negative exponential utility functions, the optimal sharing rules are linear (see, e.g., Borch [1962], Wilson [1968], Rubinstein [1974], Aase [1992, 1993b, 1993d]).

3.2. *Pricing results*

Given the model outlined above, we intend to use pricing theory for our representative agent economy. For any of the given assets in the economy having accumulated, real[8] dividends given by D^i, the market equilibrium price S^i at time t is given by

$$S^i(t) = \frac{1}{u'(X_t, t)} E\left\{ \int_t^T (u'(X_{s^-}, s^-)\, dD^i(s) + d[D^i, u'](D)) \mid \mathcal{F}_t \right\}, \quad P\text{-a.s.}, \qquad 0 \le t \le T, \tag{7}$$

where $u'(x, t) = \exp\{-\rho t - \alpha x\}$, where $[D^i, u']$ is the square covariance process. By the above remarks we can also use the pricing formula (7) for any asset in this market. We use the representative agent interpretation, where an introduction of a new instrument will change allocations and hence results in a new equilibrium. In this new equilibrium, all assets, both old and new, are priced by Eq. (7). Let

$$p_t = \exp\left(-\int_0^t r(u)\, du \right), \quad \text{and} \quad p_{t,s} = \exp\left(-\int_t^s r(u)\, du \right), \tag{8}$$

where r is the short-term world interest rate in the market, which we assume exogenously given. Consider a forward contract on the index $\hat{Z} = Z(t)/\Pi$, where $\hat{Z}(t)$ is the loss ratio index. The real accumulated dividend process D for the forward contract is given by

$$D(s) = \begin{cases} 0 & \text{when } t < s < T \\ (\hat{Z}_T - \tilde{F}_t) & \text{when } s \ge T, \end{cases} \tag{9}$$

where $\tilde{F}(t)$ is the real forward price process. We then have the following formula for $\tilde{F}(t)$:

Proposition 1: *In the above proposed model, the forward-price process is given by*

$$\tilde{F}_t = \frac{E_t\{u'(X_T, T)\hat{Z}_T\}}{E_t\{u'(X_T, T)\}} \quad \textit{for } 0 \leq t \leq T, \tag{10}$$

where E_t signifies the conditional expectation operator given $\mathcal{F}_t$.

Proof. Starting with the real accumulated dividend process D given in (9), and noticing that $\tilde{F}(t)$ by definition is determined at each time, $t < T$ in such a manner that nothing is actually paid then, the price process $S(t)$ in formula (7) is identically equal to zero for all $t < T$, which means that we get

$$0 = \frac{E_t\{u'(X_T, T)(\hat{Z}_T - \tilde{F}_t)\}}{u'(X_t, t)} \quad \text{for all } 0 \leq t \leq T, \tag{11}$$

Since the square covariance process $[D, u']$ vanishes for this D, which then gives, since $\tilde{F}_t$ is $\mathcal{F}_t$-measurable, that $\tilde{F}_t$ must be given by the formula (10), the partial equilibrium result that we sought. □

We now turn to futures contracts. Denote the futures price process of a catastrophe index by F_t, and let $\zeta_t := u'(X_t, t)/u'(X_0, 0)$. Since ζ is a state price deflator, the quantity $\xi_t := p_t^{-1}\zeta_t$ is a P-martingale, where $\xi_0 = 1$. We now have the following result:[9]

Proposition 2: *The futures price process F_t is given by*

$$F_t = \frac{E_t\left\{u'(X_T, T)p_{t,T}^{-1}\hat{Z}_T\right\}}{E_t\left\{u'(X_T, T)p_{t,T}^{-1}\right\}} \quad \text{for } 0 \leq t \leq T. \tag{12}$$

Proof. From the definition of a futures contract we know that F is the real cumulative dividend process associated with the contract. Using the pricing formula (7), by contract design the value of the contract at initiation is set to zero, so it must be the case that

$$\frac{E_t\left\{\int_t^T u'(X_{s^-}, s^-)\, dF_s + d[F, u'](s)\right\}}{u'(X_t, t)} = 0 \quad \text{for all } 0 \leq t \leq T. \tag{13}$$

In general, our jump dynamics may be expressed as follows:

$$\zeta_t = 1 + \int_0^t \int_{R_+} g^\zeta(s, y)\lambda(s)\, dH(y)\, ds + \int_0^t \int_{R_+} h^\zeta(s, y)\nu(dy; ds),$$

and

$$F_t = F_0 + \int_0^t \int_{R_+} g^F(s, y)\lambda(s)\, dH(y)\, ds + \int_0^t \int_{R_+} h^F(s, y)\nu(dy; ds).$$

Here $\nu(A, t)$ is a random measure recording the number of jumps some fundamental stochastic process makes in the time interval $(0, T]$ with values falling in the set A, λ is the frequency of jumps,[10] $H(y)$ is the cumulative probability distribution of the jump sizes having values is R_+, and the functions h and g are both predictable processes.[11]

Returning to Eq. (13), it can now be expressed in the form:[12]

$$E_t\left\{\int_t^T\left(\zeta_{s-}\,dF_s+\int_{R_+}h^F(s,y)h^\zeta(s,y)\nu(dy;ds)\right)\right\}=0,\quad 0\le t<T. \tag{14}$$

The economic contents of this equation do not change after deflation by the process $p_t^{-1}=\exp(\int_0^t r(u)\,du)$, which gives

$$E_t\left\{\int_t^T\zeta_{s-}\,p_{s-}^{-1}dF_s+\int_t^T p_s^{-1}\int_{R_+}h^F(s,y)h^\zeta(s,y)\nu(dy;ds)\right\}=0,\quad 0\le t\le T. \tag{15}$$

We now compute the term $E_t\{\xi_T\int_t^T dF_s\}$ using the product formula:

$$E_t\left\{\xi_T\int_t^T dF_s\right\}=E_t\left\{\int_t^T\xi_{s-}\,dF_s+\int_t^T(F_{s-}-F_t)\,d\xi_s\right.$$
$$\left.+\int_t^T\int_{R_+}h^\xi(s,y)h^F(s,y)\nu(dy;ds)\right\}.$$

Using the definition of ξ, and utilizing that the process $\xi_t=\zeta_t p_t^{-1}$ is a P-martingale, this is seen to be equal to[13]

$$E_t\left\{\int_t^T p_{s-}^{-1}\zeta_{s-}\,dF_s+\int_t^T p_s^{-1}\int_{R_+}h^\zeta(s,y)h^F(s,y)\nu(dy;ds)\right\}=0,$$
$$\text{for all } 0\le t\le T,$$

where the equality sign follows from (15). Thus we have that

$$E_t\left\{\zeta_T p_T^{-1}\int_t^T dF_s\right\}=0\quad\text{for all } 0\le t\le T,$$

and since F_t is $\mathcal{F}_t$-measurable, this means that $F_tE_t\{\zeta_T p_T^{-1}\}=E_t\{\zeta_T p_T^{-1}F_T\}$, which gives the conclusion of the proposition, since $p_t^{-1}=\exp\{\int_0^t r(u)\,du\}$ is also $\mathcal{F}_t$-measurable. □

Using the particular utility function defined after Eq. (7) we get for the forward contract that

$$\tilde{F}_t=\frac{E_t\left\{e^{-\alpha X_T}\hat{Z}_T\right\}}{E_t\left\{e^{-\alpha X_T}\right\}}\quad\text{for all } 0\le t\le T. \tag{16}$$

Similarly, the futures price process $F(t)$ is given by

$$F_t = \frac{E_t\left\{e^{-\alpha X_T} p_{t,T}^{-1} \hat{Z}(T)\right\}}{E_t\left\{e^{-\alpha X_T} p_{t,T}^{-1}\right\}} \quad \text{for all } 0 \leq t \leq T. \tag{17}$$

Since we consider a particular insurance market, it seems reasonable to assume that the short-term interest rate r is conditionally statistically independent of Z and W, given $\mathcal{F}_t$, in which case $\tilde{F}(t) = F(t)$ for all $t \leq T$, an assumption we shall adopt from here on.

In Section 4 we carry out the calculation indicated in these equations for our model for the index Z.

The condition that ξ is a P-martingale will put some restriction on the premium rate function a_t. It is easy to see that if we want to restrict attention to cases where a_t is a constant a, this means that the accumulated consumption process W must have homogeneous increments.

Before we leave this section and move to the main results in Sections 4 and 5, we will briefly indicate how the equivalent martingale measure looks like in the present context.

3.3. *An interpretation of the market price of insurance risk*

Starting with the stochastic model we have chosen, the Radon-Nikodym derivative $\xi(T) = \frac{dQ}{dP}$ and its associated density process $\xi(t) = E(\xi(T) \mid \mathcal{F}_t)$ can alternatively be represented as

$$\xi(t) = \left(\prod_{n \geq 1} \kappa v(Y_n) 1(\tau_n \leq t)\right) \exp\left\{\int_0^t \int_R (1 - \kappa v(y)) \lambda H(dy)\, ds\right\}, \tag{18}$$

holding for any $t \in [0, T]$. Here $H(dy)$ is the cumulative distribution function of the claim sizes Y_1, κ is a positive constant and $v(y)$ is a function satisfying

$$\int_0^\infty v(y) H(dy) = 1. \tag{19}$$

The term $v(y)H(dy)$ is accordingly the probability distribution of the claim sizes Y_1 under the probability measure Q, whereas $\kappa\lambda$ is the corresponding frequency of the Poisson process under Q. We interpret κ as the market price of frequency risk, and $v(y)$ as the market price of claim size risk. The function v is nonnegative and strictly positive on the support of H.

To get a feeling why ξ_t is a P-martingale, consider the simple case where all the Y_k are equal to 1—that is, the case where the index is simply a Poisson process. In this case $\xi_t = \kappa^{N_t} \exp\{(1-\kappa)\lambda t\}$, and it is easy to show that $E(\xi_s \mid \xi_t) = \xi_t$ for $t \leq s$ by, for example, using the moment generating function of the Poisson random variable.

In more general models the parameter κ and the function v and H will depend on both t and possibly also on the state $\omega \in \Omega$. For any smooth enough utility function u for the representative agent so that we can use Ito's generalized differentiation formula on the term

$\ln\{u'(X(t), t)\}$, we may compare terms in the relation $u'(X(t), t)/u'(X(0), 0) = p_t \xi(t)$, where $\xi(t)$ is given in (19). After some calculations we obtain

$$\kappa v(y) = \frac{u'(X(t) - y, t)}{u'(X(t), t)}, \quad \text{for } 0 \leq t \leq T \quad \text{and} \quad y \geq 0. \tag{20}$$

Under risk aversion we notice that the product $\kappa v(y) > 1$ for all $y > 0$, and since $v(y)$ is a density satisfying (20), it follows that $\kappa > 1$. *Thus, in a risk-averse insurance market, the risk-adjusted frequency $\lambda\kappa$ of the claims is larger than the* (*objective*) *frequency* λ. In particular for our representative agent (21) reduces to

$$\kappa v(y) = e^{\alpha y}, \quad \text{for } 0 \leq t \leq T \quad \text{and} \quad y \geq 0. \tag{21}$$

We notice from this expression that when the insurance claim size y increases, the term $\kappa v(y)$ increases, which seems reasonable in view of our interpretation of the terms κ and v. Furthermore, observe that κv increases as the absolute risk aversion α of the market increases.

Using standard properties of the gamma (n, μ)-distribution, from (20), it follows that

$$\kappa = \left(\frac{\mu}{\mu - \alpha}\right)^n, \tag{22}$$

and by our assumptions $\alpha < \mu$, so we see that $\kappa > 1$. It also follows from (22) that the claim sizes Y_j are distributed as gamma $(n, \mu - \alpha)$ under Q. They are also independent under Q, so that Z is indeed a compound Poisson process having frequency $\kappa\lambda$ and claim size distribution gamma $(n, \mu - \alpha)$ under Q (see, e.g., Delbaen and Haezendonck [1989] and Aase [1993d]).

We notice that once we have a representative agent, this agent determines the equivalent martingale measure Q uniquely, and the probability distribution of the claims index Z, or the aggregate net reserves X, will depend on this representative agent, here, for example, through the parameter α.

A final point: Even if our chosen utility function u is unbounded below, we are still on solid ground, since expected utility essentially amounts to taking expectations under Q, which still leaves us in the gamma class of probability distributions having moments of all orders.

4. Catastrophe futures contracts: A basic futures formula

4.1. Introduction

In this section we apply the results of the previous sections to the pricing of futures contracts on insurance indexes specified in our model.[14] In the next section we return to contracts actually offered in the CBOT market. In the present case the contract that used to be offered was more a future derivative $\phi(F_T)$, where ϕ is a function of the form $\phi(F_T) = \$25{,}000 \min(\hat{Z}_T, 2)$—that is, a cap with cap-of-point 2, where $F_T = \hat{Z}_T$ by the principle of convergence in futures markets. In Section 5 we also consider futures call options and spreads.

4.2. *A simple futures pricing formula*

For the model we have chosen, and under the independence assumptions for the short rate r, we want to compute the expression on the right-hand side of Eq. (17) or what amounts to the same under interest rate independence, of Eq. (16). We make the simplifying assumption that the premium rate $a_t \equiv a > 0$, a restriction we discuss briefly after the proof of the following theorem:

Theorem 1: *Under the assumptions above, the following futures formula obtains*

$$F_t = \hat{Z}_t + \frac{n\mu^n \lambda(T-t)}{\Pi(\mu-\alpha)^{n+1}} \quad \textit{for } t \leq T \quad \textit{and} \quad 0 \leq \alpha < \mu. \tag{23}$$

Proof. We must compute

$$F(t) = \frac{E\left\{\left(\exp\left[-\alpha\left(W(T) + aT - \sum_{i=1}^{N(T)} Y_i\right)\right]\right)\hat{Z}(T) \,\middle|\, \mathcal{F}_t\right\}}{E\left\{\exp\left[-\alpha\left(W(T) + aT - \sum_{i=1}^{N(T)} Y_i\right)\right] \,\middle|\, \mathcal{F}_t\right\}} \quad \text{for } t \leq T. \tag{24}$$

To this end we start by considering the denominator of (24). Defining the functions $h(\alpha, t) := E\{\exp(\alpha(W(T) - W(t))) \mid \mathcal{F}_t\}$ and $g(\alpha) := E\{\exp(\alpha Y_1)\} = (\frac{\mu}{\mu-\alpha})^n$, the latter equality following from the distributional assumptions on Y_1, we get that

$$\begin{aligned}
&E\left\{\exp\left[-\alpha\left(W(T) + aT - \sum_{i=1}^{N_T} Y_i\right)\right] \,\middle|\, \mathcal{F}_t\right\} \\
&\quad = h(\alpha, t)e^{-\alpha X(t) - \alpha a(T-t)} E\left\{g(\alpha)^{N_T - N_t} \,\middle|\, \mathcal{F}_t\right\} \\
&\quad = h(\alpha, t)e^{-\alpha X(t) - \alpha a(T-t)} E\left\{g(\alpha)^{N_{T-t}}\right\} \\
&\quad = h(\alpha, t)\exp\{-\alpha X(t) - \alpha a(T-t) + \lambda(T-t)(g(\alpha) - 1)\}.
\end{aligned} \tag{25}$$

Here we have used the properties of the compound Poisson process Z, all the independence structure, its independent increments and time homogeneity, and its independence from the process W.

Moving to the numerator of (24), we again use that the compound Poisson process Z has independent increments and is time homogeneous. Let us denote by H_i the cumulative probability distribution function of Y_i and note that under our assumptions all the H_i are equal—to H, say. Also we denote the convolution of H by itself k times by H^{*k}. Thus we get:

$$
\begin{aligned}
&E\left\{\left(\exp\left[-\alpha(W(T)+aT-\sum_{i=1}^{N(T)}Y_i\right]\right)\sum_{i=1}^{N(T)}Y_i\,\middle|\,\mathcal{F}_t\right\}\\
&=h(\alpha,t)e^{-\alpha X(t)-\alpha a(T-t)}E\left\{e^{\alpha\sum_{i=N(t)+1}^{N(T)}Y_i}\left(Z_t+\sum_{i=N(t)+1}^{N(T)}Y_i\right)\,\middle|\,\mathcal{F}_t\right\}\\
&=h(\alpha,t)e^{-\alpha X(t)-\alpha a(T-t)}E\left\{e^{\alpha(Z_T-Z_t)}(Z_t+(Z_T-Z_t))\,\middle|\,\mathcal{F}_t\right\}\\
&=h(\alpha,t)e^{-\alpha X(t)-\alpha a(T-t)}\left(Z_tE\left\{e^{\alpha Z_{T-t}}\right\}+E\left\{e^{\alpha Z_{T-t}}Z_{T-t}\right\}\right)\\
&=h(\alpha,t)e^{-\alpha X(t)-\alpha a(T-t)}\\
&\left\{Z_t\sum_{k=0}^{\infty}\frac{e^{-\lambda(T-t)}(\lambda(T-t))^k}{k!}\int_{[0,\infty)}\cdots\int_{[0,\infty)}e^{\alpha\sum_{i=1}^{k}y_i}dH_1\ldots dH_k\right.\\
&\left.+\sum_{k=1}^{\infty}\frac{e^{-\lambda(T-t)}(\lambda(T-t))^k}{k!}\int_{[0,\infty)}\cdots\int_{[0,\infty)}e^{\alpha\sum_{i=1}^{k}y_i}\sum_{i=1}^{k}y_i\,dH_1\ldots dH_k\right\}\\
&=h(\alpha,t)e^{-\alpha X(t)-\alpha a(T-t)}\\
&\left\{Z_t\sum_{k=0}^{\infty}\frac{e^{-\lambda(T-t)}(\lambda(T-t))^k}{k!}\int_{[0,\infty)}e^{\alpha v}dH^{*k}(v)\right.\\
&\left.+\sum_{k=1}^{\infty}\frac{e^{-\lambda(T-t)}(\lambda(T-t))^k}{k!}\int_{[0,\infty)}e^{\alpha v}v\,dH^{*k}(v)\right\}.
\end{aligned}
$$

Note that zero catastrophes in $(t, T]$ occurs with probability $e^{-\lambda(T-t)}$, accounting for the first term in the sum multiplying Z_t. By using the usual convention that terms of the type $\sum_{i=1}^{k} y_k = 0$ when $k = 0$, the first term in the second sum is zero, so we start it at $k = 1$ instead of $k = 0$.

Now, using the same reasoning as for the numerator, the term corresponding to the first sum above equals

$$Z_t h(\alpha, t)\exp\{-\alpha X(t) - \alpha a(T-t) + \lambda(T-t)(g(\alpha) - 1)\}.$$

Since H is the distribution function of the gamma (n, μ), the k-fold convolution H^{*k} is the distribution function of a gamma (kn, μ), and we get that

$$
\begin{aligned}
\int_{[0,\infty)}e^{\alpha v}v\,dH^{*k}(v)&=\int_0^{\infty}e^{\alpha v}v\frac{\mu^{kn}}{\Gamma(kn)}v^{kn-1}e^{-\mu v}\,dv\\
&=\frac{kn}{\mu}\int_0^{\infty}e^{\alpha v}\frac{\mu^{kn+1}}{\Gamma(kn+1)}v^{(kn+1)-1}e^{-\mu v}dv=\frac{kn}{\mu}\left(\frac{\mu}{\mu-\alpha}\right)^{kn+1}.
\end{aligned}
$$

Thus we obtain for the term corresponding to the second sum

$$h(\alpha, t)\exp\{-\alpha X(t) - \alpha a(T-t)\}\sum_{k=1}^{\infty}\frac{e^{-\lambda(T-t)}(\lambda(T-t))^k}{k!}\int_{[0,\infty)} e^{\alpha v} v dH^{*k}(v)$$
$$= h(\alpha, t)\frac{n}{\mu}\left(\frac{\mu}{\mu-\alpha}\right)^{n+1}\lambda(T-t)\exp\left\{-\alpha X(t) - \alpha a(T-t) + \lambda(T-t)\left(\left(\frac{\mu}{\mu-\alpha}\right)^n - 1\right)\right\}.$$

We now have the following expression for the numerator in (24):

$$\left[Z_t + \frac{n\mu^n\lambda(T-t)}{(\mu-\alpha)^{n+1}}\right]h(\alpha, t)\exp\left\{-\alpha X(t) - \alpha a(T-t) + \lambda(T-t)\left(\left(\frac{\mu}{\mu-\alpha}\right)^n - 1\right)\right\}.$$

Combining this with the expression (25) for the denominator, the formula (23) results, after dividing the numerator by Π.[15] □

It is easy to see that the proof goes through if a_t is time varying and also a random process, as long as it is independent of Z_t. However, for ζ to be a state price deflator this means that $\exp\{-\rho t - \alpha X_t\}p_t^{-1}$ is a P-martingale, and thus a restriction must hold on the premium rate a:

$$a = \frac{1}{\alpha}\left[\lambda(g(\alpha) - 1) - \rho + \frac{1}{T-t}\ln\{h(\alpha, t)\} + \frac{1}{T-t}\ln\{E_t(p_{t,T})\}\right], \quad \text{all } t \leq T. \tag{26}$$

From this condition we notice that the cumulant generating function of W must factor out the term $(T-t)$, which is indeed a very common property of, for example, Levy-processes. The restriction on the interest-rate process r is trivially satisfied when r is a constant but is in general restrictive for stochastic interest rates. However, this assumption can be avoided if we relax the requirement that $a_t \equiv a$, in which case the equilibrium relation (26) will change accordingly.

4.3. Discussion of the futures formula

Notice that the formula (23) depends only on parameters that can be estimated from market data. In particular, this is true for the process parameters n, μ, and λ, but also α may be estimated from available price data. The formula (23) can be written as

$$F_t = \hat{Z}_t + \frac{n\lambda(T-t)}{\Pi\mu}\left(\frac{\mu}{\mu-\alpha}\right)^{n+1} = \hat{Z}_t + E(\hat{Z}_T - \hat{Z}_t \mid \mathcal{F}_t)\left(\frac{\mu}{\mu-\alpha}\right)^{n+1},$$

the last expression being true since $E(\hat{Z}_T - \hat{Z}_t \mid \mathcal{F}_t) = \lambda(T-t)\frac{n}{\Pi\mu}$. We remark the following:

F_t depends on preferences in the market through the parameter α, the intertemporal coefficient of absolute risk aversion in the market. The futures price process F_t is seen to increase as α increases, ceteris paribus—that is, more risk aversion in the market leads to more expensive "reinsurance" as well as to higher required risk premiums for the investors/speculators on the opposite side of the contracts. This seems to be well in accordance with intuition.

F_t increases, ceteris paribus, with the time to settlement $(T-t)$, with the claims frequency λ, and finally, as the expected sizes of the claims increase through the parameter n (remember that $EY_1 = n/\mu$).

F_t does not depend on the aggregate consumption W in the market. This is a consequence of the fact that for the exponential utility, the absolute risk aversion is not depending on the level of wealth. This is, of course, one unrealistic feature of our model, since it would indeed be desirable with some kind of consumption dependence on F_t (even if this quantity is difficult to estimate).

4.4. *Risk premiums*

Implicit in our model lies that the market, not the actuaries, will decide the futures prices and hence the risk premiums. The risk premium in insurance economics is defined as the difference between the market price of a contract and the expected payout under the contract. In our model the risk premium equals

$$F_t - \{\hat{Z}_t + E(\hat{Z}_T - \hat{Z}_t) \mid \mathcal{F}_t)\} = \frac{n}{aT\mu}\lambda(T-t)\left\{\left(\frac{\mu}{\mu-\alpha}\right)^{n+1} - 1\right\}. \tag{27}$$

Since $\alpha < \mu$, the term $(\frac{\mu}{\mu-\alpha})^{n+1} > 1$ for n a positive integer, so the risk premium is positive, in accordance with our earlier remarks. Under risk neutrality $\alpha = 0$, in which case it follows from (27) that the risk premium is zero. Thus the term $(\frac{\mu}{\mu-\alpha})^{n+1}$ corrects for risk aversion in the market. For a given value of μ, the risk premium is an increasing function of both α and n, and for given values of α and n, the risk premium is a *decreasing* function of μ. Since $EY_1 = n/\mu$ and $\operatorname{var} Y_1 = n/\mu^2$, an increase in μ leads to a decrease in both the expected size of the loss and in the variance of the loss, so as a result a risk-averse market would tend to require less compensations for risk bearing, also well in accordance with economic intuition.

4.5. *Risk-adjusted evaluation*

Returning to the results of Section 3.3, we may compute $F(t) = E^Q(\hat{Z}_T \mid \mathcal{F}_t)$ by alternatively finding the distribution of Z under the equivalent martingale measure Q, which in our model we have seen correspond to $\kappa \cdot v(y) = e^{\alpha y}$, where κ equals the risk adjustment on the frequency λ. Let $H(dy) = h(y)dy$, so that $h(y)$ is the probability density of the gamma (n, μ)-distribution. Since $\int v(y)h(y)\,dy = 1$ from condition (13), we see that $\kappa \int v(y)h(y)\,dy = (\frac{\mu}{\mu-\alpha})^n$, $\alpha < \mu$, so $\kappa = (\frac{\mu}{\mu-\alpha})^n > 1$. Thus the risk adjusted frequency equals $\lambda\kappa > \lambda$, as mentioned before.

Furthermore the claim sizes $Y_1, Y_2, \ldots$ are all independent and identically distributed with density $v(y)h(y)$, meaning that they are all gamma $(n, \mu - \alpha)$-distributed. From the comments in the above subsection, since $\alpha > 0$, *this risk adjustment amounts to making the claim size distribution more risky under Q than under P*. Thus, in the constructed Q-economy where the "pseudo-agents" are all risk neutral, they agree on a probability distribution for the loss ratio index that is more risky than the objective distribution.

After all this, the above computation is straightforward, since Z is a compound Poisson process also under Q and $F(t) = E^Q(\hat{Z} \mid \mathcal{F}_t) = \hat{Z}_t + E^Q(\hat{Z}_T - \hat{Z}_t \mid \mathcal{F}_t) = \hat{Z}_t + \lambda \cdot \kappa(T - t)\frac{n}{\mu-\alpha}\frac{1}{\Pi}$, which is exactly the formula (23). Thus we here have an alternative proof of Theorem 1.

In conclusion, formulas (23), and (27) for the market value of the insurance catastrophe futures contract and the associated economic risk premium seem to capture well some essential economic features of the pricing of the risky instruments under investigation.

In the final section of the article we turn to the pricing of derivatives on the future index.

Before we end this section, we illustrate the use of catastrophe insurance futures by the following simple numerical example.

Example 1. Ins Ltd. expects to earn \$1 million in premiums on its insurance policies during the third quarter of 1995. Ins Ltd.'s actuaries have forecasted that \$950,000 in catastrophic losses will be incurred by the company for this period. It is also predicted that the third-quarter catastrophic losses for the sample companies that report to ISO will be \$25,000,049, with associated premiums Π amounting to \$26,417,200. Ins Ltd. decides to buy December 1995 Eastern Catastrophe contracts at the beginning of July. Ignoring possible reporting lags, we assume that the key parameters are estimated as follows: $\hat{\lambda} = 10$, $(T - t) = 0.25$, $\hat{n} = 10$, $\hat{\mu} = 10^{-6}$, $(\hat{\mu}/(\hat{\mu} - \hat{\alpha}))^{11} = 1.05669$, resulting in $\hat{a} = 5 \cdot 10^{-9}$. Here $E(\hat{Z}_T) = 0.946355$. According to the formula (23) the futures price equals \$25,000. The company decides to buy 40 contracts (=1 million/25,000).

Scenario A. Assume that the weather in August and September was worse than anticipated. Ins Ltd.'s actual catastrophic losses turn out to be \$1,900,000, twice as much as anticipated. The ISO reporting companies were similarly affected, and the total catastrophic losses incurred by these companies during the third quarter were \$50,000,098. The final settlement price then becomes $\$25{,}000 \cdot 2 = \$50{,}000$. The gain from 40 long futures contracts then becomes $\$(50{,}000 - 25{,}000) \cdot 40 = \$1$ million, more than offsetting Ins Ltd.'s unexpected catastrophic losses.

Scenario B. Assume that the weather in the third quarter turned out better than forecasted. Ins Ltd. lost only \$570,000 (one half of the anticipated amount), the ISO reporting companies lost \$15 million. In this case the final settlement price was \$15,000. In the case that Ins Ltd. held the futures position long until settlement, the loss would amount to \$400,000 on this position, resulting in \$20,000 less than anticipated in the final result. On these contracts the investors/speculators on the opposite side of the contracts made \$400,000.

5. Derivatives on the futures index

5.1. Introduction

In practice futures contracts are not the frequently traded instruments in CBOT market but rather spreads on the futures index. Usually insurers are accustomed to profit from losses that are less than originally anticipated. As an alternative the insurer can, for example, buy a call option on the futures index.

Furthermore it is likely that a cap is needed to limit the credit risk in the case of unusually large losses, and this will also have the advantage of making the contract behave more like a nonproportional reinsurance policy. This contract is therefore really a futures derivative with price at maturity $\phi(F_T)$, where $F_T = \hat{Z}_T$, and it is characterized in the CBOT market as follows:

$$\phi(F_T) = \$25{,}000 \min\left(\frac{Z(T)}{\Pi}, 2\right) = \$25{,}000\{\hat{Z}(T) - \max(\hat{Z}(T) - 2, 0)\}, \quad (28)$$

where $\hat{Z}(T) = Z(T)/aT$ is the loss ratio. Thus, the settlement value is the trading unit (\$25,000) times the loss ratio, capped at an amount equal to twice the trading unit. This is equivalent to a long position in the loss ratio plus a short position in a call option on the loss ratio with a strike price of 2. Below we shall find the market value of this contract.

We make a distinction here between a *pure futures option* and a *conventional futures option*. A conventional call option, for example, requires payment of the option premium when purchased and at exercise pays the buyer any excess of the underlying asset price over the exercise price. A pure futures option, on the other hand, calls for the buyer to receive (or pay) daily any change in the futures option price in order to mark the buyer's margin account to market. For our representative utility function $u'(x, t) = \exp\{-\alpha x - \rho t\}$, by our results in Section 3 it follows that the equilibrium market price of a pure futures contract in our model then becomes

$$\pi^{\phi}(F_t, t) = \frac{E_t\left\{u'(X_T, T)p_{t,T}^{-1}\phi(F_T)\right\}}{E_t\left\{u'(X_T, T)p_{t,T}^{-1}\right\}} \quad \text{for } 0 \le t \le T, \quad (29)$$

while that of a conventional contract is

$$\pi_c^{\phi}(F_t, t) = \frac{E_t\{u'(X_T, T)\phi(F_T)\}}{u'(X_t, t)} \quad \text{for } 0 \le t \le T. \quad (30)$$

The connection between the price of the conventional and the pure futures instrument is as follows. Consider the price $\pi_f^{\phi}(F_t, t)$ of a forward contract on the underlying futures instrument. It is

$$\pi_f^{\phi}(F_t, t) = \frac{E_t\{u'(X_T, T)\phi(F_T)\}}{E_t\{u'(X_T, T)\}} \quad \text{for } t \le T. \quad (31)$$

Denote by

$$\Lambda_{t,T} := \frac{E_t\{u'(X_T, T)\}}{u'(X_T, T)} \quad \text{for } t \leq T.$$

It follows that

$$\pi_f^{\phi}(F_t, t) = \frac{1}{\Lambda_{t,T}} \pi_c^{\phi}(F_t, t). \tag{32}$$

We notice that under the before mentioned independence assumption related to the interest rate r, $\pi_f^{\phi}(F_t, t) = \pi^{\phi}(F_t, t)$. Thus under these assumptions $\pi^{\phi}(F_t, t) = \frac{1}{\Lambda_{t,T}} \pi_c^{\phi}(F_t, t)$, so if $\Lambda_{t,T} \leq 1$, which must hold true if the interest rate r is positive, a pure futures instrument has a price higher than or equal to that of a conventional futures contract.

5.2. Prices of futures derivatives for our fully specified model

We consider our model of Section 3, where Z is the compound Poisson process and the intertemporal representative utility function is the negative exponential. We retain the conditions of Theorem 1. Consider a pure futures instrument with terminal payoff $\phi(x)$ on the futures index x. The market premium is denoted by $\pi^{\phi}(F_t, t)$, whereas the market premium of the corresponding conventional futures instrument is denoted by $\pi_c^{\phi}(F_t, t)$. Let us continue to assume the conditional interest rate independence of r from X and Z. We then have under the above assumptions

Theorem 2: *The market price of the pure futures instrument is given by*

$$\pi^{\phi}(F_t, t) = e^{-\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n} \left\{ \phi(\hat{Z}_t) + \sum_{k=1}^{\infty} \frac{\left(\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right)^k}{k!} E\{\phi(\hat{Z}_t + \hat{V}_k) \mid \hat{Z}_t\} \right\}, \tag{33}$$

where $\hat{V}_k$ is conditionally gamma $(kn,\ aT(\mu-\alpha))$-distributed, given $\hat{Z}_t$.

The market price of the conventional futures instrument is

$$\pi_c^{\phi}(F_t, t) = h(\alpha, t) e^{-(\lambda+\rho+a\alpha)(T-t)} \times \left\{ \phi(\hat{Z}_t) + \sum_{k=1}^{\infty} \frac{\left(\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right)^k}{k!} E\{\phi(\hat{Z}_t + \hat{V}_k) \mid \hat{Z}_t\} \right\}. \tag{34}$$

Proof. Starting with the pure futures instrument, the market price is given by

$$\pi^{\phi}(F_t, t) = \frac{E\left\{e^{-\alpha(W(T)+aT-Z(T))} \phi(\hat{Z}_T) \mid \mathcal{F}_t\right\}}{E\left\{\exp\left[-\alpha\left(W(T) + aT - \sum_{i=1}^{N(T)} Y_i\right)\right] \mid \mathcal{F}_t\right\}}. \tag{35}$$

The denominator in (35) is given by the expression

$$\exp\{-\alpha X(t) - \alpha a(T-t) + \lambda(T-t)(g(\alpha)-1)\}h(\alpha, t),$$

where $g(\alpha) = (\frac{\mu}{\mu-\alpha})^n$ and $h(\alpha, t) = E\{\exp(\alpha(W(T) - W(t))) \mid \mathcal{F}_t\}$. As for the numerator, we can use the same line of reasoning as in the proof of Theorem 1. This leads to

$$\begin{aligned}
&E\left\{e^{-\alpha(W(T)+aT-Z(T))}\phi(\hat{Z}_T) \,\middle|\, \mathcal{F}_t\right\} \\
&= h(\alpha, t)e^{-\alpha X(t)-\alpha a(T-t)}E\left\{e^{\alpha(Z_T-Z_t)}\phi(\hat{Z}_t + (\hat{Z}_T - \hat{Z}_t)) \,\middle|\, \mathcal{F}_t\right\} \\
&= h(\alpha, t)e^{-\alpha X(t)-\alpha a(T-t)} \\
&\quad\times \left\{\phi(\hat{Z}_t)e^{-\lambda(T-t)} + \sum_{k=1}^{\infty} e^{-\lambda(T-t)}\frac{(\lambda(T-t))^k}{k!}\int_{[0,\infty)} e^{\alpha v}\phi(\hat{Z}_t + \hat{v})H^{*k}(dv)\right\},
\end{aligned}$$

where $\hat{v} = v/aT$. The first term in the parenthesis is the contribution if no accidents happen in the time interval $(t, T]$, in which case the size of the additional claims equals zero, an event that happens with probability $\exp\{-\lambda(T-t)\}$. The convolution integral can be written

$$\begin{aligned}
&\int_{[0,\infty)} e^{\alpha v}\phi(\hat{Z}_t + \hat{v})F^{*k}(dv) \\
&= \int_0^{\infty} e^{\alpha v}\phi(\hat{Z}_t + \hat{v})\frac{\mu^{kn}}{\Gamma(kn)}v^{kn-1}e^{-\mu v}dv \\
&= \left(\frac{\mu}{\mu-\alpha}\right)^{kn}\int_0^{\infty}\phi(\hat{Z}_t + \hat{v})\frac{(\mu-\alpha)^{kn}}{\Gamma(kn)}v^{kn-1}e^{-(\mu-\alpha)v}dv.
\end{aligned}$$

The last integral is the conditional expected value of $\phi(\hat{Z}_t + \hat{V}_k)$ given $\hat{Z}_t$, where $\hat{V}_k = V_k/aT$ and the conditional distribution of V_k given $\hat{Z}_t$ is gamma $(kn, \mu-\alpha)$.[16] Thus we obtain (33).[17]

To prove the second part, we first compute the discount factor $\Lambda_{t,T}$. Using the results in the proof of Theorem 1, we get directly that

$$\begin{aligned}
\Lambda_{t,T} &= \frac{E_t\{u'(X_T, T)\}}{u'(X_t, t)} \\
&= h(\alpha, t)\exp\{-(\rho + \lambda + a\alpha)(T-t) + \lambda(T-t)g(\alpha)\}. \qquad (36)
\end{aligned}$$

The expression (34) now follows from (32), (33), and (36), using the conditional interest-rate independence mentioned earlier. □

We notice that the futures price for a pure futures instrument does not depend on the impatience rate ρ in the market, but the option price connected to a conventional futures contract does depend on ρ.

Formulas (33) and (34) may be taken as the starting point for deriving useful pricing formulas for futures derivatives in practice. One would expect that numerical techniques must be employed, but we are indeed able to derive closed form expressions and approximations below.

5.3. A futures cap

As an important illustration of Theorem 2, let us consider a cap where $\phi(\hat{Z}_T) = \$25{,}000 \times \min(\hat{Z}_T, c)$. In ordinary reinsurance this contract exhibits similar characteristics to a non-proportional reinsurance treaty with an upper limit. We concentrate on the pure futures derivative, since the conventional market price just differs by some multiplicative constant, given the information at time t. We consider the case where $\hat{Z}_t < c$.

Theorem 3: *Assume that $\hat{Z}_t < c$. Then the real market price at time t of the futures cap with expiration T is given by*

$$\pi^{(x\wedge c)}(F_t, t) = \$25{,}000\Bigg[F_t + \exp\left\{-(\mu-\alpha)\Pi(c-\hat{Z}_t) - \lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right\} \times\left((c-\hat{Z}_t)\Sigma_1^c - \frac{n}{aT\mu}\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^{n+1}\Sigma_0^c\right)\Bigg], \tag{37}$$

where

$$\Sigma_0^c = \sum_{k=0}^{\infty}\frac{\left(\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right)^k}{k!}e_{(k+1)n}((\mu-\alpha)\Pi(c-\hat{Z}_t)), \tag{38}$$

$$\Sigma_1^c = \sum_{k=1}^{\infty}\frac{\left(\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right)^k}{k!}e_{kn-1}((\mu-\alpha)\Pi(c-\hat{Z}_t)), \tag{39}$$

and where

$$e_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}. \tag{40}$$

Proof. Let $\min(x, y) = (x \wedge y)$. If $\hat{Z}_t \geq c$, obviously $\pi^{(x\wedge c)}(F_t, t) = \$25{,}000c$, so let us consider the interesting case where $\hat{Z}_t < c$. According to Theorem 2 we then have to compute

$$E\{(\hat{Z}_t + \hat{V}_k)\wedge c \mid \hat{Z}_t\} = \hat{Z}_t + E\{\hat{V}_k \wedge (c-\hat{Z}_t) \mid \hat{Z}_t\}$$
$$= \hat{Z}_t + \int_0^{\infty}\min(\hat{v}, c-\hat{Z}_t)\frac{(\mu-\alpha)^{kn}}{\Gamma(kn)}v^{kn-1}e^{-(\mu-\alpha)v}dv$$

$$= \hat{Z}_t + \frac{kn}{aT(\mu - \alpha)} \Gamma(kn + 1, (\mu - \alpha)\Pi(c - \hat{Z}_t))$$
$$+ (c - \hat{Z}_t)(1 - \Gamma(kn, (\mu - \alpha)\Pi(c - \hat{Z}_t))),$$

where $\Gamma(n, \mu x)$ is the cumulative probability distribution function of the Gamma (n, μ)-distribution, called the incomplete gamma function. Since n is an integer, we have the following relation

$$\Gamma(n, \mu x) = 1 - e_{n-1}(\mu x) e^{-x\mu} \tag{41}$$

(see, e.g., Abramowitz and Stegun [1972]). Using (40) we obtain

$$E\{(\hat{Z}_t + \hat{V}_k) \wedge c \mid \hat{Z}_t\}$$
$$= \hat{Z}_t + \frac{kn}{aT(\mu - \alpha)} \big(1 - e_{kn}((\mu - \alpha)\Pi(c - \hat{Z}_t)) e^{-(\mu-\alpha)\Pi(c - \hat{Z}_t)}\big)$$
$$+ (c - \hat{Z}_t) e_{kn-1}((\mu - \alpha)\Pi(c - \hat{Z}_t)) e^{-(\mu-\alpha)\Pi(c-\hat{Z}_t)}.$$

Inserting this into (33) we get

$$\pi^{(x \wedge c)}(F_t, t) = \$25{,}000 e^{-\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n}$$
$$\times \Bigg\{ \hat{Z}_t + \sum_{k=1}^{\infty} \frac{\left(\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right)^k}{k!}$$
$$\times \left(\hat{Z}_t + \frac{kn}{aT(\mu - \alpha)} \big(1 - e_{kn}((\mu - \alpha)\Pi(c - \hat{Z}_t)) e^{-(\mu-\alpha)\Pi(c-\hat{Z}_t)}\big)\right.$$
$$\left. + (c - \hat{Z}_t) e_{kn-1}((\mu - \alpha)\Pi(c - \hat{Z}_t)) e^{-(\mu-\alpha)\Pi(c-\hat{Z}_t)} \right) \Bigg\}.$$

Consider now the series. The first two terms are

$$\hat{Z}_t \big(e^{\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n} - 1\big) + \frac{n}{aT(\mu - \alpha)} \lambda(T - t) \left(\frac{\mu}{\mu - \alpha}\right)^n e^{\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n},$$

and the last two terms can be written

$$-\frac{n}{aT(\mu - \alpha)} \lambda(T - t) \left(\frac{\mu}{\mu - \alpha}\right)^n e^{-(\mu-\alpha)\Pi(c-\hat{Z}_t)}$$
$$\times \sum_{k=0}^{\infty} \frac{\left(\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right)^k}{k!} e_{(k+1)n}((\mu - \alpha)\Pi(c - \hat{Z}_t))$$
$$+ (c - \hat{Z}_t) e^{-(\mu-\alpha)\Pi(c-\hat{Z}_t)} \sum_{k=1}^{\infty} \frac{\left(\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right)^k}{k!} e_{kn-1}((\mu - \alpha)\Pi(c - \hat{Z}_t)).$$

Using (38), (39), and Theorem 1, we now obtain the conclusion of the theorem. □

The two last terms in (37) adjust the futures price F_t given in (23) for the capping off at the $c \cdot 100\%$ point of the loss ratio.

The above formula is fairly simple, and may sometimes be further simplified by observing that the sum in (40) converges quickly to e^x, so that for n large or moderately large we may substitute the exponential for this truncated sum. Assume now that this approximation is reasonable. We shall comment on the error we are doing below. In this case we get the approximations

$$\Sigma_0^c \approx \exp\left\{(\mu - \alpha)\Pi(c - \hat{Z}_t) + \lambda(T - t)\left(\frac{\mu}{\mu - \alpha}\right)^n\right\} \tag{42}$$

and

$$\Sigma_1^c \approx \exp\left\{(\mu - \alpha)\Pi(c - \hat{Z}_t) + \lambda(T - t)\left(\frac{\mu}{\mu - \alpha}\right)^n\right\} - \exp(\mu - \alpha)\Pi(c - \hat{Z}_t), \tag{43}$$

where the given formulas are both upper bounds. Inserting these expressions into (37) we get the approximation

$$\pi^{(x \wedge c)}(F_t, t) \approx \begin{cases} 25000\left[c + (\hat{Z}_t - c)\exp\left\{-\lambda(T - t)\left(\frac{\mu}{\mu - \alpha}\right)^n\right\}\right] & \text{if } \hat{Z}_t < c \\ 25000c & \text{if } \hat{Z}_t \geq c. \end{cases} \tag{44}$$

This formula may violate the simple arbitrage constraint for certain values of the parameters, in which case one must consider the smaller of F_t and the above expression.

We see that the futures cap price approaches \$25,000c from below as $\hat{Z}_t$ approaches c from below. In the case $\hat{Z}_t < c$, we observe that as the risk parameter μ increases, the futures cap price decreases. Furthermore, the market price in (44) is seen to be an increasing function of the claim frequency parameter λ, of the claims size parameter n, of the intertemporal absolute risk aversion parameter α of the market and of the time to settlement $(T - t)$.

5.4. *The bound of the approximation error*

Here we give a bound on the approximations (42) and (43). Let us use the notation $x = \lambda(T - t)(\frac{\mu}{\mu - \alpha})^n$ and $y = (\mu - \alpha)\Pi(c - \hat{Z}_t)$, where both x and y are seen to be positive. Given positive constants $K > 0$ and $\varepsilon > 0$ there exists an integer n_0 such that if $|x| \leq K$, $|y| \leq K$ and $n \geq n_0$ then

$$\left|e^{x+y} - \Sigma_0^c\right| < e^K \frac{(K + \varepsilon)^{n+1}}{(n + 1)!} \quad \text{and} \quad \left|(e^x - 1)e^y - \Sigma_1^c\right| < e^K \frac{(K + \varepsilon)^{n+1}}{(n + 1)!},$$

which tells us that the approximations we have done are good for n even of moderate size, due to the rapid increase of factorials. Since the bounds of the above sums are both upper bounds, and since these sums appear with opposite signs in all our pricing formulas, the final error is further reduced. However, in certain ranges of the parameter values the approximation is poor—for example, when the term y gets large, which means when Z_t is small relative to c.

5.5. Futures call options

In this subsection we compute the market price of futures call options. This contract mimics to some extent a standard stop loss reinsurance treaty. Again we only treat the pure futures version. Let us denote the market value of the pure futures call option by $\pi^{(x-c)^+}(F_t, t)$, where c now stands for the call exercise price, where $\phi(x) = (x - c)^+ = \max(x - c, 0)$. Assuming the futures call option has the same expiration date as the underlying futures contract, according to Theorem 2 we have to compute

$$\pi^{(x-c)^+}(F_t, t) = \$25{,}000 e^{-\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n} \times \left\{ \phi(\hat{Z}_t) + \sum_{k=1}^{\infty} \frac{\left(\lambda(T-t)\left(\frac{\mu}{\mu-\alpha}\right)^n\right)^k}{k!} E\{(\hat{Z}_t + \hat{V}_k - c)^+ \mid \hat{Z}_t\} \right\}.$$

This quantity can now readily be found for the interesting case $\hat{Z}_t < c$, from the results in the previous section, Theorem 3 and the use of (28). For the case $\hat{Z}_t > c$ we immediately obtain $\pi^{(x-c)^+}(F_t, t) = \$25{,}000(F_t - c)$, which can, for example, also be proved, as a check, using (35) and the increasing property of Z. For $\hat{Z}_t < c$, we now get

$$\pi^{(x-c)^+}(F_t, t) = \$25{,}000 \left[\exp\left\{ -(\mu - \alpha)\Pi(c - \hat{Z}_t) - \lambda(T - t)\left(\frac{\mu}{\mu - \alpha}\right)^n \right\} \right] \times \left(\frac{n}{aT\mu} \lambda(T - t) \left(\frac{\mu}{\mu - \alpha} \right)^{n+1} \Sigma_0^c - (c - \hat{Z}_t)\Sigma_1^c \right). \tag{45}$$

From our two expressions for the market price for the futures call option, we see that $\pi^{(x-c)^+}(F_t, t) \to \$25{,}000(\hat{Z}_t - c)^+$ as $t \to T$, where we have to remember that the process $\hat{Z}$ is a nondecreasing process in the time parameter.

If we can use the above approximation, we get

$$\pi^{(x-c)^+}(F_t, t) \approx \begin{cases} 25000\left[(\hat{Z}_t - c)\left(1 - \exp\left\{-\lambda(T - t)\left(\frac{\mu}{\mu - \alpha}\right)^n\right\}\right) + \frac{n}{\Pi\mu}\lambda(T - t)\left(\frac{\mu}{\mu - \alpha}\right)^{n+1}\right] & \text{if } \hat{Z}_t < c \\ 25000(F_t - c) & \text{if } \hat{Z}_t \geq c. \end{cases} \tag{46}$$

The simple arbitrage restrictions $F_t \geq \pi^{(x-c)^+}(F_t, t) \geq \max\{0, F_t - c\}$ may be violated for this approximation.

We notice that the futures call option decreases as the strike price c increases, and it increases as $\hat{Z}_t$ increases, as normally is the case for a call option. Naturally, when c grows large compared to the other parameter values, the approximation becomes poor and eventually invalid, since the price can become negative because of the linearity in c. A closer examination of the proof of Theorem 3 reveals why this is so.

Example 2. Consider again Ins Ltd. in Example 1 in the same market structure. Instead of buying futures contracts, the company decides to buy call options on the futures index with strike price $c = 1.75$ (corresponding to $\$43{,}740 = \$25{,}000 \cdot 1.75$). Using the exact formula (45) we obtain $\pi^{(x-1.75)^+}(F_0, 0) = \$2{,}598$. Still buying 40 contracts, the immediate call option cost equals $103,920.

Scenario A: In this case the final settlement price equals $50,000, so Ins Ltd.'s total option position settled at a market value equal to $250,000, an overall market net gain of $146,080, offsetting 15 percent of its $950,000 in unexpected incurred losses, clearly a catastrophe protection; here less effective than the futures position of Example 1.

Scenario B: In this case the final settlement price equals $15,000, and with a strike price of $43,740, the gain equals zero, resulting in an overall market loss of $103,920. In this case the company made $380,000 more than anticipated, resulting in net $276,080 more than expected. This is to be compared to the futures position in Example 1, leading to a net result of $20,000 less than anticipated, illustrating the advantages to the buyer of call options over that of a buyer of futures contracts in a scenario with losses less than projected. In the present scenario the insurer keeps his upside potential.

5.6. A call spread

The final contract we consider is a call spread, which may be thought of as a capped futures call option. By this we mean a futures instrument having the payoff

$$\phi(x) = \$25{,}000 \begin{cases} 0 & \text{if } x \leq c_1 \\ (x - c_1) & \text{if } c_1 < x \leq c_2 \\ (c_2 - c_1) & \text{if } x > c_2. \end{cases} \tag{47}$$

The capping of the call option will limit the risk of the investors on the opposite side of the insurers at the exchange. This contract looks very much like a conventional nonproportional reinsurance treaty of the $X.L.$-type, with a "retention" c_1 and an upper limit c_2, where $c_1 < c_2$.

The market price of this contract follows from the above results, since to hold such a contract is equivalent to hold long one futures call option with strike price c_1 and to sell short one futures call option with strike price c_2. This follows since the payoffs at expiration are identical for these two positions. Thus

$$\pi^{\phi}(F_t, t) = \pi^{(x-c_1)^+}(F_t, t) - \pi^{(x-c_2)^+}(F_t, t), \tag{48}$$

where the two expressions to the right are found from Eq. (45) above. This is the market price of a bull spread. For the simplified version in (46), we get an approximation of the market value of the contract (47) as

$$\pi^{\phi}(F_t, t) \approx \$25{,}000 \begin{cases} \left(1 - \exp\left\{-\lambda(T-t)\left(\dfrac{\mu}{\mu-\alpha}\right)^n\right\}\right)(c_2 - c_1), & \text{if } \hat{Z}_t < c_1 \\ (c_2 - c_1) + (\hat{Z}_t - c_2) \\ \quad \times \exp\left\{-\lambda(T-t)\left(\dfrac{\mu}{\mu-\alpha}\right)^n\right\}, & \text{if } c_1 \leq \hat{Z}_t < c_2 \\ (c_2 - c_1), & \text{if } \hat{Z}_t \geq c_2 \end{cases} \tag{49}$$

—a simple formula. Again we must impose the simple arbitrage restrictions, since they may be violated by this approximation. The most interesting case is $\hat{Z}_t < c_1$, where we see that the price *increases* with the difference $(c_2 - c_1)$ between the upper limit and the "retention" limit, the claim frequency parameter λ, time to settlement $(T - t)$, the market parameter α of risk attitude, and with the risk parameter n of the loss ratio index. The market price is seen to be a *decreasing* function of the risk parameter μ, which seems natural in view of the fact that increasing this parameter decreases the "riskyness," properly interpreted, of the loss ratio index. In the case $c_1 < \hat{Z}_t < c_2$, we notice that the price increases with the value of $\hat{Z}_t$, and otherwise we get the same signs in the comparative statics analysis for the parameters λ, $(T - t)$, n, μ, and α as in the first case. It is not likely that this approximation is very accurate, however, especially in the region $\hat{Z}_t < c_1$, when $\hat{Z}_t$ is small. The approximation improves as the index increases. Otherwise, the same comment about the validity of this approximation could be remarked here as for the formula (46).

The formulas in this section remain valid also in the case where there is no underlying futures index that is traded, since we have used the principle of convergence $F(T) = \hat{Z}(T)$, and only considered futures instruments with the same expiry data T as the futures contract. Thus only a loss ratio index is really needed, making the analysis relevant also for PCS contracts. In the latter case we must, of course, use Z_T instead of $\hat{Z}_T$.

Example 3. Continuing the two previous examples, suppose Ins Ltd. buys a 1.60/1.80 spread, typically a catastrophe protection. Using the exact formula (48), the market price equals $\pi^{\phi}(F_t, t) = \$972$, leading to a spread cost of \$38,880 on 40 contracts. The gain on these contracts in Scenario A equals $\$5{,}000 \cdot 40 = \$200{,}000$, resulting in a net gain of \$161,112, which is better than in the case of the 1.75-options of Example 2. In Scenario B the gain is net \$341,200 more than expected, also a better result than for the call options.

6. Conclusion

We have presented a valuation model of forward and futures contracts and of futures derivatives, when the underlying asset is an accumulated insurance loss ratio. We have taken into account the essentials in modeling such an index by a relevant nondecreasing stochastic process containing claim size jumps at random time points of accidents.

Equilibrium market premiums are derived of futures prices and of futures derivatives. In particular, we have presented closed-form formulas for the futures price, futures call options, futures caps, and capped futures call options—the most important contracts traded on the CBOT insurance futures exchange. From these the prices of most other futures instruments can readily be derived. We have suggested an approximation to simplify the formulas for the futures derivatives, making the expressions tractable for comparative statics and analytic treatment. However, this approximation may be inaccurate at the beginning of the trading quarter, where it may violate simple arbitrage restrictions.

The theory is in a form where it may easily be tested from market data. This may reveal if we have gone too far in our simplification. Parsimony is important in theoretical analyses, but realism is important in practice. It will be interesting to see how well the values derivable from this article will fit observed prices in this insurance market. We will also get an estimate of the risk attitude in this market. The article gives directions toward a more general theory, however, from which even more realistic approaches could start. This is not likely to lead to closed-form solutions, but numerical techniques must then be employed.

Using statistical inference for stochastic processes on the data from the CBOT exchange, we intend to test our results empirically, in particular the simple formulas. If more realism is needed, we will try to handle that as well, within the framework of this article in a subsequent investigation.

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Notes

1. The analysis is not restricted to CAT products only; PCS options are also included.
2. In general they would seek a perfect hedge for the excess claims beyond a retention level.
3. Many institutional details and facts can be found in the bibliography at the end.
4. One could argue that in aggregate consumption the premiums are in zero net supply, in which case this can be taken care of by the term W_t. In any case, our results turn out not to depend upon either of these terms.
5. See below for the definition of $\mathcal{F}_t$.
6. When $N(t) = 0$, the convention is that the random sum $Z(t)$ is set to zero.
7. If linear spanning were possible for the futures contracts, then the price process would not depend on the particular choice of equivalent martingale measure.
8. Real prices refer to prices relative to the price of the consumption commodity.

9. Our results for forward and futures contracts are consistent with the results of Richard and Sundaresan [1981] for a continuous time, Markov diffusion model, but our method of proof is very different. They base their proof on the infinitesimal generator of the Markov process and use a Feynman-Kac type approach.
10. The intensity process λ may be both stochastic (predictable) and time dependent.
11. For more details, see, e.g., Aase [1992].
12. Here we employ the results on dynamic equilibria for jump processes in Aase [1993a].
13. Technically we must here assume a certain integrability of F. This may be verified in our model if we, for example, assume that the short interest rate r is bounded, since the formula (17) is unique, $u'(\cdot)$ is decreasing and bounded on $[0, \infty)$ and Z_T is L^2.
14. CAT-futures were traded for two years at the CBOT, are not traded at the present but may start trading again at some time in the future, in the United States or, perhaps, in Europe.
15. The numerator could alternatively be found by differentiating the denominator with respect to α, involving considerably less work, but this does unfortunately not generalize to the contracts to be studied in Section 5.
16. The variables $\hat{Z}_t$ and V_k are thus independent.
17. As a check, it is easy to verify that formula (33) agrees with the basic futures formula (23) when $\phi(x) = x$ for all x.

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