CHAPTER 3

Dynamics (finite discrete models)

In this chapter we generalize the previous model to an intertemporal framework. Here we will assume that the horizon is finite, T, and that time is discrete $t = 0, .1 \dots T$. In the same way, uncertainty is modelled through finite sets of states of nature. In this framework we will derive the main result : the price of an asset is equal to the expected present discounted value of its payments. The expectation being computed with some "risk neutral" probability.

3.1. The tree of states of nature

We will model uncertainty as following. At date 0 there is one unique state of nature. At date 1 several states of nature are possible

$E_1 = \left\{e_1^1, \cdots e_1^{N_1}\right\}$, for each state of nature e_1 of E_1, several sates of nature are possible at date 2, and so on up to T. Call E_t the set of possible states of nature at date t . We can define a "successor" relation between these states of nature.

Definition 19. At each date the set of states of nature is E_t. We say that e_{t+1} in E_{t+1} is a successor of e_t in E_t, $e_{t+1} > e_t$, if e_{t+1} is one of the possible states at date $t+1$ if the realized state of nature at date t is e_t.

In a tree structure, for all t (except $t = 0$) , each state of nature is the sucessor of one unique state of nature. We have hence the following lemma :

Lemma 20. *for all t and t' , $t < t'$, for all state $e_{t'}$ there exists a unique state e_t at t such that there is a successor path between e_t and $e_{t'}$. (Past is perfectly known!).*

We can define partial information on this tree structure. The idea is that if one has some information, this information cannot be forgotten.

Definition 21. Information and Filtration. An information structure $\mathcal{P}_t$ at date t is a partition of E_t. There is perfect information if $\mathcal{P}_t$ is composed by all the singletons $\left\{e_t^i\right\}$. There is no information if $\mathcal{P}_t = \{E_t\}$.

We say moreover that $(\mathcal{P}_t)_{t=0 \dots T}$ is a filtration if, for all t, all A_t in $\mathcal{P}_t$, all e_t and e'_t in A_t the predecessors of e_t and e'_t belongs to the same set in the partition $\mathcal{P}_{t-1}$.

On this tree structure we can define a process simply as a mapping that takes values at the different states of nature.

DEFINITION 22. A process X is a mapping that takes real values (potentially mulidimensional) on the tree. We note $X_t(e_t)$ the value of the process at date t in the state e_t.

EXAMPLE 23. In the Cox, Ross Rubinstein Model at each date, the price of a stock can be multiplied either by u or or by d . $S_t = \varepsilon.S_{t-1}$ with $\varepsilon = d$ or u. Here a state of nature at date t is given by a sequence on t digits $e_t = (\varepsilon_1, \varepsilon_2, \cdots \varepsilon_t)$ each digit being d or u. There are exactly 2^t states of nature at date t.

3.2. Stochastic process on a tree

We can add a probability structure on the tree. There are two differents ways to do so. The first one consists in defining, for each state e_t the probability transition $\pi(./e_t)$ defined in E_{t+1} :

DEFINITION 24. For each e_t in E_t the probability transition $\pi(./e_t) \geq 0$ gives the probability of each state in E_{t+1} knowing that the realized state at t is e_t.

$\pi(e_{t+1}/e_t) = 0$ if e_{t+1} is not a successor of e_t.

$\sum_{e_{t+1} > e_t} \pi(e_{t+1}/e_t) = 1$

It is then very easy to compute the probability of any state $e_{t'}$ at any anterior date in any predecessor state.

LEMMA 25. *The probability $\pi(e_{t'}/e_t)$ where $t' > t$ and e_t the unique state at t which is predecessor of $e_{t'}$ is :*

$\pi(e_{t'}/e_t) = \pi(e_{t'}/e_{t'-1}) \times \pi(e_{t'-1}/e_{t'-2}) \times \cdots \times \pi(e_{t+1}/e_t)$ *where* $e_{t'} > e_{t'-1} > e_{t'-2} > \cdots > e_{t+1} > e_t$ *is the unique backward path from* $e_{t'}$ *to* e_t.

The "ex ante" (at date 0) probability of a state is hence :

$\pi(e_t) = \pi(e_t/e_0) = \pi(e_t/e_{t-1}) \times \cdots \times \pi(e_1/e_0)$ *where* $e_t > e_{t-1} > \cdots > e_1 > e_0$ *is the unique backward path from* e_t *to* e_0.

The other way is to give the probability of the terminal states $\pi(e_T)$. Then the probability of any state $\pi(e_t)$ is simply the probability of subset of terminal states that are successors of e_T :

$$\pi(e_t) = \sum_{e_T > e_t} \pi(e_t)$$

3.3. No arbitrage condition on a dynamic model

In this paragraph we want to derive the no arbitrage condition. We suppose that there exists K assets. These assets are defined by their dividend process $d_i(e_t)$ which gives for each sate e_t at date t the cash flow the owner earns. We simply define a portfolio strategy as a mapping that gives at each date and state the quantity of assets owned in the portfolio. These quantity can be negative in the sense that short selling is allowed. The no arbitrage condition will give a condition on prices.

Definition 26. We note $\theta_i(e_t)$ the quantity of asset i owned if the state is e_t. The price of asset i in the state e_t is noted $p_i(e_t)$.

To a portfolio strategy is associated a cash flow and a value.

The cash flow is simply composed of dividends and the net product of transactions :

$$W_\theta(e_t) = \sum_i \left(\theta_i(e_{t-1})d_i(e_t) + p_i(e_t)\left(\theta_i(e_{t-1}) - \theta_i(e_t)\right)\right)$$

Where e_{t-1} is the unique predecessor of e_t .

At date 0 this gives $W_\theta(e_0) = -\sum_i p_i(e_0)\theta(e_0)$, which is simply the opposite of the initial cost of the portfolio.

The value of the portfolio is :

$$V_\theta(e_t) = \sum_i p_i(e_t)\theta_i(e_t)$$

We have obviously :

$$W_\theta(e_t) + V_\theta(e_t) = \sum_i \theta_i(e_{t-1})\left[d_i(e_t) + p_i(e_t)\right]$$

The no arbitrage condition imposes some restriction on the process of prices p. Indeed we have :

Definition 27. An arbitrage portfolio is a portfolio such that $W_\theta(e_t) \geq 0$ for all t and e_t with at least one strict inequality. An arbitrage portfolio makes money without taking any risk.

We have the following proposition :

Proposition 28. *The market is arbitrage free if there exists a set of "pseudo state transition prices" $q(e_t/e_{t-1}) \geq 0$ such that for all asset i :*

$$p_i(e_t) = \sum_{e_{t+1} > e_t} q(e_{t+1}/e_t)\left[d_i(e_{t+1}) + p_i(e_{t+1})\right]$$

or for any portfolio :

$$V_\theta(e_t) = \sum_{e_{t+1} > e_t} q(e_{t+1}/e_t)\left[W_\theta(e_{t+1}) + V_\theta(e_{t+1})\right]$$

This proposition is quite simple to proove. It is sufficient to consider one state e_t and its successors at a given date t . Then consider portfolios such that only $\theta_i(e_t)$ is not zero. . Apply then the result obtained in the static model.

The above equation tells us that the price of an asset is equal to a weighted sum of its liquidation value at the next date. The positive weights are common to all asets.

3.4. Risk neutral probability

Assume now that there is are risk free assets. At each date, a risk free asset gives at next date a constant cash flow $d(e_{t+1}) = 1$, and is replaced by a new risk free asset. In other words, there are T short term risk free assets of maturity one. The price at date t in the state e_t of this asset (Bond) is hence :

$$B(t, e_t, 1) = \sum_{e_{t+1} > e_t} q(e_{t+1}/e_t) = \frac{1}{1 + r(e_t)}$$

Where $r(e_t)$ is simply the short term interest rate at date t in the state e_t .

Just then define :

$$\pi'(e_{t+1}/e_t) = q(e_{t+1}/e_t)\,(1 + r(e_t))$$

This is a transition probability! Since it is positive and the sum on the successors of e_t is one. This gives hence :

$$p_i(e_t) = \frac{1}{1 + r(e_t)} \sum_{e_{t+1} > e_t} \pi'(e_{t+1}/e_t)\left[d_i(e_{t+1}) + p_i(e_{t+1})\right] = \frac{1}{1 + r(e_t)} \mathrm{E}_{\pi'}\left[d_i(e_{t+1}) + p_i(e_{t+1})/e_t\right]$$

The price is simply the present value of the expected liquidation cash flow (computed with the risk neutral probability).

PROPOSITION 29. *when there exists a risk free asset, the price of any asset is the present value of its expected liquidation cash-flow, computed with some risk neutral probability. If the market is complete this probability is unique. If it is incomplete it is not, but gives the same value to all possible portfolios.*

DEFINITION 30. Consider $Q'(e_t) = \frac{1}{1+r(e_t)} \times \frac{1}{1+r(e_{t-1})} \cdots \frac{1}{1+r(e_0)}$. Where $e_t > e_{t-1} \cdots e_0$ is the unique path from e_t to e_0 . It is called the discount factor at state e_t. Let $\hat{p}_i(e_t) = Q'(e_{t-1}) p_i(e_t)$ and $\hat{d}_i(e_t) = Q'(e_{t-1}) d_i(e_t)$ the discounted price and divided of the asset i.

It is easy to compute the price of any asset at any state, function only of future dividends. Indeed we have :

$$\hat{p}_i(e_t) = \sum_{e_{t+1} > e_t} \pi'(e_{t+1}/e_t) \left[\widehat{d}_i(e_{t+1}) + \hat{p}_i(e_{t+1}) \right]$$

Define then as previously the probability of state e_t knowing e_{t_0} by : $\pi'(e_t/e_{t_0}) = \pi'(e_t/e_{t-1})\pi'(e_{t-1}/e_{t-2}) \cdots \pi'(e_{t_0+1}/e_{t_0})$, with $e_t > e_{t-1} > e_{t-2} \cdots > e_{t_0+1} > e_{t_0}$ the unique path from e_t back to e_{t_0} . We have the following results :

PROPOSITION 31. *The (discounted) price at date t_0 of an asset in the state e_{t_0} is equal to the expected discounted value of future dividends :*

$$\widehat{p}_i(e_{t_0}) = \sum_t \sum_{e_t} \pi'(e_t/e_{t_0}) \widehat{d}_i(e_t)$$

If an asset does not distibute dividends (except at final date T) then its discounted price is a martingale under the risk neutral probability:

$$\hat{p}(e_t) = \sum_{e_{t+1} > e_t} \pi'(e_{t+1}/e_t) \left[\hat{p}(e_{t+1}) \right]$$

EXAMPLE 32. The Cox Ross Rubinstein model.

Assume, in the CRR model, that $S_1(u) = uS_0$ and $S_1(d) = dS_0$ where u and d are two positive numbers with $u > d$. Assume furthermore that the short term interest rate is constant. Then the transition prices are very simple whatever the state e_t we have :

$$q(u/e_t) = q(u) = \frac{1}{1+r} \frac{1+r-d}{u-d}$$

$$q(d/e_t) = q(d) = \frac{1}{1+r} \frac{u-(1+r)}{u-d}$$

And the risk neutral probability :

$$\pi'(u) = \frac{1+r-d}{u-d}$$

$$\pi'(d) = \frac{u-(1+r)}{u-d}$$