CHAPTER 1

Introduction

The aim of this course is to give some general concepts that found the main models of finance. This in order to first better understand the jungle of financial products and second to understand the functionning of the markets.The purpose of a model is to give a representation of the reality that allows to understand phenomena and to make predictions.

We begin by some naïve questions. Because simple interrogations are often necessary to understand clearly what we talk about.

1.1. Questions

The first question is quite simple : what are financial products designed for?

1.1.1. Real position. The original purpose of financial assets is to transfer wealth through time or, alternatively or simultaneously, to share risk to mitigate it.

The idea is very simple. The economic activity of an individual can lead, to him, to a particular income stream (a so-called real position). This flow can eg be distributed over time in some special and uneven way, or be conditioned by some random events. For example, a seller of glouttes on an international market, knows his income two months after delivery, at a price that depends on the price of some foreign currency at that date. He would like to insure his income immediately, so as to be sure of the amount of money he will get in two months. He his ready to exchange a random cash flow against a sure one. To do so he buys, on a financial market or over the counter (OTC), an asset that does the job : take the risk. Obviously this is not free. But the individual feels adantageous to buy this asset.

Demand for financial assets is hence, at least initially, triggered by a need of risk coverage or transfer wealth over time. But this coverage, or transfer, has itself a price, that can be fixed, for instance on a market on which supply and demand are confronted.

We can hence summarize :

- A financial product first allows to exchange wealth through time : I would like to have one euro on the 24th december (because I'd like to buy a gift for my grand-son). What does it cost today? Here I don't want a bottle of whisky, I want one "euro on the 24th december". The "euro on the

24th december" is a particular good which is different of the "euro on the 1st november" (which is useful to buy grave flowers). These two goods have not the same price today.

- A financial product also allows to exchange risk. I know that I will have a lot of euros if it rains next week (because I sell umbrellas), and nothing if not (I don't sell sun hats). I am ready to pay something today to smoothe my wealth and have some money even when the sun shines.

1.1.2. Market makers. These definitions insist on the real initial position of the demander. But some other agents intervene on this market although they have no initial real position. Their only role is to make the market liquid. To be sure that any demand (justified by a real position), meets an offer. They are called "market makers", in the sense that they supply the demand at a price they define. The price of this supply is obviously strategic.

In some sense, banks, insurance and other institutions are "financial intermediaries" : they offer counterparts to demands (and supplies) originated by real positions.

One purpose of models is to provide methods to "price" financial assets in order the market makers can offer a counterpart to agents.

One first goal of this course is to describe model that allows to "price" assets, that is to find formulas that can give the price of a given asset.

Obviously, we will also describe models that explain how prices endogeneously arise.

1.1.3. Models and collateral questions. We are going to study several types of models. We can split them in two families :

- the arbitrage models : in these models we use a a particular condition of price equilibrium, the no arbitrage condition. Under this condition we can derive a very important property on the price structure.
- the behaviour models : in the behaviour models, we try to endogeneize prices. Here we will focus on several hypothesis on individual expectations and on individual information.

The different models can be static, dynamic, discrete and continuous.

Other questions are addressed in this course. One, for instance, is the question of market efficiency, and, information revelation through prices. The question is, as soon as some people are better informed, is that true that this information is immediately transfered to price so that, initially uninformed people are, by so, finally informed.

In the same way we will try to study models that could explain speculative bubbles.

For the first lesson we just describe some characteristics of the market and give one very simple model that can help us to understand complex ones.

1.2. Assets

Financial instruments can be decomposed in three classes :

- debt assets : the issuer of the paper promises a given future cash flow to the holder.
- equity assets : the paper represents a share of a company and gives right to potential (and not known in advance) dividends.
- currencies which are traded on the foreign exchange.

On the top of these basic "papers" one can imagine derivative assets.

- A derivative asset simply gives conditional cash flows that depend on the value of the underlying asset. These derivatives can be traded on a market or on an "Over The Counter" (OTC) basis.

The following table gathers the main different assets :

Asset Class	Instrument type		
	Securities	Exchange-trade derivatives	OTC derivatives
Debt (LT>1year)	Bonds	Bond options or futures	Interest rate swap, options
Debt (ST≤1year)	Bills, Commercial paper	Short term interest rate futures	forward rate agreements
Equity	Stock	Stock options, equity futures	stock options, exotic
FX (foreign exchange)	-	futures	swaps

To these assets, it important to add the markets of commodities (gold, oil, sugar, wheat...) which were the first ones to involve futures.

1.2.1. Equity assets : Stocks. A share is a title of ownership of a share of the capital of a company. At issuance, the value of these securities is determined by society. Then, the securities are traded on a market and the price is set according to supply and demand.

In addition, the holder receives a share of company profits as dividends. hence, from a pure financial viewpoint, a stock is a financial paper that gives right to receive, in the future, cash flows corresponding to the firm profits. Mathematically this amounts to a promise of random future cash flows.

Definition 1. (Notation) A stock is an asset which gives right to dividends $\widetilde{d}(t)$ or ($d(t,\omega)$, where ω is the state of nature) at time t. Stocks are traded on an Exchange platform. Supply an demand determine prices.

- Euronext is the European exchange created in September 2000 by the merger of managing markets stock exchanges of Paris, Brussels (BXS) and Amsterdam (AEX), Stock Exchanges of Lisbon and Porto (BVLP) have been included in this group in 2001, LIFFE (London International Financial Futures and Options Exchange) in 2002. In June 2006, Euronext announced a merger agreement with the NYSE (New York Stock Exchange), merger became effective in April 2007. Nyse-Euronext is a most important trading platform in the world.

- Indexes

An index is generally used to follow the evolution of prices of stocks. An index is generally the cumulated value of all the stocks of a given set of companies.

Most global stock markets offer indexes. The main French stock market indexes are:

- The CAC 40 is the main index of the Paris market. It is calculated from the prices of 40 stocks selected from hundred companies generating a high volume of trade on Euronext Paris.

- The CAC Next 20 includes the next twenty companies.

- The SBF 120 is calculated from the stocks of the CAC 40 and CAC Next 20 and 60 stocks of the first and second market.

- The SBF 250 calculated from 250 companies of all sectors. It is composed of the CAC 40 and CAC Next 20 and the CAC Mid 100 and CAC Small 90.

Obviously let us quote the well known international indexes :

- The Dow Jones Industrial Average. It is the oldest index in the world. It is composed of 30 major industrial stocks of the New York Stock Exchange (blue chips). For historical reasons, its value is the arithmetic average (not weighted by capitalization) assets that compose it.

- The SP 500 lists the 500 largest companies on Wall Street. Its value is more representative than the Dow Jones.

- The Nasdaq Composite Index is an index calculated from all the values of NASDAQ (second equity markets of the United States after NYSE). It contains more than 300 assets, but are not limited to technology.

- The Nikkei 225 index of leading the Tokyo Stock Exchange. Its value is calculated as the Dow-Jones.

- The Footsie (FTSE 100) is the main index of the London Stock Exchange.

- The DAX is the main index of the Frankfurt Stock Exchange consists of 30 blue chip stocks.

1.2.2. Debt : Bonds. A bond is a negotiable instrument issued by a corporation or a public authority for a loan. A Bond is defined by the name of the issuer, interest rate, maturity date, the currency in which it is issued, a periodicity of coupon, the date of issue. Repayment terms and the method of payment of the lenders are contractually fixed, the remuneration may be fixed or variable (indexed to the rate of interest and not on the outcome of the company).

From a mathematical point of view, a fixed rate bond is an asset that promise sure future cash flows.

Definition 2. (Notation) a Bond is associated to a loan issued (by a borrower) at date t_0 for a length (maturity) T. It is defined by the sequence of future "sure" cash flows (for the holder) $d(t)$ (coupons) and, when T is finite, a final payment $h(t_0 + T)$, (in this case $t > T + t_0 \Rightarrow d(t) = 0$) .

- "ultimately" or "In fine" bonds are defined by a face value (nominal or principal) $N$and a nominal rate r_0 per period such that $d(t) = N * r_0$ for $t < t_0 + T$ and $h(t_0 + T) = N(1 + r_0)$.

- Constant annuities are bonds such that $h(t_0 + T) = d(t)$

- A perpetuity is a bond for which $T = +\infty$.

- The nominal zero-coupon at date t_0 with maturity T is bond is such that $d(t) = 0$ and $h(t_0 + T) = N = 1$

Bonds are exchanged on the market. Bonds heve different names according to their maturity :

- short term (bills): maturities between one to five year; (instruments with maturities less than one year are called Money Market Instruments)
- medium term (notes): maturities between six to twelve years;
- long term (bonds): maturities greater than twelve years.

Variable rate bonds, are on contrary a promise of "random" cash flows contingent to some interest rate.

Nominal interest rate, nominal value, and conditions of repayment hence allows the calculation of coupon that the borrower agrees to pay annually (usually in the euro area), quarterly (especially English and American bonds) or with a shorter periodicity. Bond prices are expressed in percentage of the nominal value. Thus, a bond par value 10,000 euros (price cut) is not priced 9900 euros, but 99%. Even if, theoretically, a fixed rate bond is sure, there is a risk of default of the issuer. So, since the bond market is being internationalized, investors need ratings to measure the risk of issuer default.

States issue bonds in order to raise funds on the markets. The main European bonds are OAT (France), Bonos (Spain), Olo (Belgium), Btp (Italy) Guilt (UK), Bund (Germany).

1.2.3. Swap. A swap is a contract in which two counterparties agree to exchange two sets of cash flows (usually debt or currency) between two dates. Unlike trade in financial assets, trade in financial flows are instruments dealed "Over The Counter" OTC without affecting the balance sheet. The synthetic product resulting from the exchange reflects the characteristics sought by the investor. It amounts to accessing a synthetic product not available on the regular market. For example, two parties may agree to pay at predetermined times the difference between a fixed rate and a variable rate.

1.2.4. Forwards and futures. A futures contract is a commitment to buy or sell at a certain maturity T an amount of an asset (securities or commodities) negotiated at the date of commitment, but payable at maturity T .

Definition 3. The forward contract negociated at t_0 for a term T sets at date t_0 a forward price $f(t_0, T)$ for a given quantity of a given good. Hence if $p(t)$ is the spot price at date t of this good, the seller of the forward contract will earn $f(t_0, T) - p(T)$ at T payed by the buyer.

Then we see that a forward contract involves a final payment from one party to the other which amounts to the difference between spot price and forward price.

Futures contracts are traded on regulated markets as opposed to forwards, which are traded "over the counter". The underlying Futures are either physical assets (futures commodity having existed since antiquity) or financial assets (such futures appeared in the early 1970s). Regulated markets (such as the Chicago Mercantile Exchange Group, NYSE Euronext LIFFE) offer standardized futures contracts with respect to the amounts, timing and quality of the underlying assets. They have clearinghouses that serve as intermediaries, so that the buyer has in front of him these clearinghouses as a seller, and vice versa for the seller. At any date the difference $f(t-1,T)-f(t,T)$ is payed by the buyer to the seller. Daily payments between winners and losers are made exclusively through the clearinghouse. Counterparty risk being transferred, the guarantee of futures contracts is thus much more important against the risk of default by the losing party than for contracts OTC (Forwards for which the loss is disbursed at maturity). Futures on physical assets are futures contracts whose underlying assets are commodities : agricultural products (beef, pork, dairy, wheat, mas, soybean, sweet, wood, etc.). , metals (gold, silver, copper, aluminum, zinc, palladium, etc..) and matters relating to energy (gas, oil, coal, electricity). Futures on these physical assets, which enable producers and traders to hedge against the risk of price change, most often give rise to the delivery of the underlying commodity unlike financial assets.

1.2.5. Options.

Definition 4. An option is a contract between two parties and allow one of the parties to ensure, on payment of a premium, the right (but not the obligation) to buy or sell to the other party a particular asset at a predetermined price at the end of a certain period (called European options) or during a certain period (American options). The underlying asset can be a financial asset (stock, bond, treasury bond, futures, currencies, indices, etc..) or a physical asset (agricultural or mineral). The value of the option is the amount of the premium that the option buyer is willing to pay the seller. An option is said to be negotiable if it can be traded on a regulated market. Otherwise, one speaks of OTC trading.

1.2.6. Structured Products. A structured product is a product designed by a bank to meet the needs of its customers. This is often a complex combination of conventional and derivatives. This will, for example, a fixed-rate investment with participation rising or falling prices of a basket of shares. They can take various legal forms and are traded OTC. As a structured product can not be traded on a market, its price is determined using mathematical models reproduces the behavior of the product over time and different market trends. These are often products with high margins.

1.2.7. Foreign exchange. In the foreign exchange market, currencies (money) are exchanged.

1.3. Functioning of trading

- Orders

When you want to buy or sell an asset you submit an order. There are roughly speaking two kinds of orders : no limit orders and limit price orders. No limit means that you are ready to sell or buy, whatever the price. Limit order means that you set a ceiling (cap) price for buying or a floor price for selling. These orders are ranked in the following way : highest buy orders are put on the top of the list of buy orders, and are ranked decreasingly. Lowest sell orders are put on the top of sell orders and are ranked increasingly.

Call (v_i, b_i) the ith buy order (ready to buy a quantity b_i for a price less than v_i) and (c_j, s_j) the jth sell order . We have $v_1 \geq v_2 \geq ...v_i \geq v_{i+1}$and $c_1 \leq c_2 \leq ... \leq c_j \leq c_{j+1}$ ($v_1 = +\infty$ in case of no limit buy order, and $c_1 = -\infty$ in case of no limit sell order)

Practically these orders are gathered in the "order book" and ranked in the previous way.

- Trading session

There are three periods : pre-opening and fixing, trading session, closing and fixing. During pre-opening orders are gathered in the "order book".

1.3.1. Order book and opening fixing.

An order book gathers sell and buy orders, when the market is closed the order book accumulates orders : high price buy orders and low price sell orders are placed on the top.

Consider for instance this (fictious) order book (real order books involve prices with 2 or even 3 decimals). On the left are buy orders, on the right sell orders.

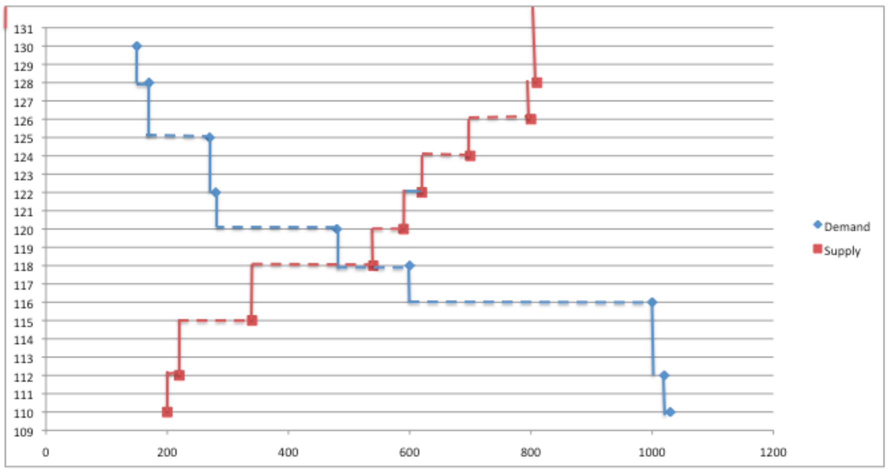
In the following order book the second buy order is a limit price order : ready to buy 50 units at a (cap) price 130.

Cumulative	Quantity	Price	Price	Quantity	Cumulative
100	100	no limit	no limit	150	150
150	50	130	110	50	200
170	20	128	112	20	220
270	100	125	115	120	340
280	10	122	118	200	540
480	200	120	120	50	590
600	120	118	122	30	620
1000	400	116	124	80	700
1020	20	112	126	100	800
1030	10	110	128	10	810

The equilibrium price is 118 : at this price buyers are ready to buy 600 units and sellers 540. That means that only 540 units will be effectively sold at 118. After the fixing, the order book is as follows. Top orders are executed and remain at 118, 60 to buy.

Cumulative	Quantity	Price	Price	Quantity	Cumulative
60	60	118	120	50	50
460	400	116	122	30	80
480	20	112	124	80	160
490	10	110	126	100	260
			128	10	270

On can draw supply/demand curves very easily.



1.3.2. Intraday and closing. Then orders are executed (if there is a counterpart) as they come in the order book.

1.3.3. Market makers. Some agents have no initial real position. They intervene on the market only as "counterparty" to be sure that "real position" agents will find a counterparty to trade with.

Hence a so called market maker is a company, or an individual, that quotes both a buy and a sell price in a financial instrument. For a given asset a market maker sets hence two prices :

- Ask : the (floor) price they are ready to sell the asset.
- Bid : the price (ceiling) they are ready to buy.

Ask price is larger than bid price. One says that there is a "bid-ask spread". A market maker hence presents simultaneously a supply and a demand curve. Both are horizontal and the demand curve is below the supply one.

There are several factors that contribute to the difference between the bid and ask prices. The most evident factor is a security's liquidity. This refers to the volume or amount of stocks that are traded on a daily basis. Some stocks are traded regularly, while others are only traded a few times a day. The stocks and indexes that have large trading volumes will have narrower bid-ask spreads than those that are infrequently traded. When a stock has a low trading volume, it is considered illiquid because it is not easily converted to cash.

Another explanation of the bid-ask spread Bid ask spread is the asymmetry of information among traders. This is detailed with an example of microstructure models (Glosten-Milgrom). (see chapter on behaviour models).

1.4. Two first models

1.4.1. The miracle of complete and perfect markets : the risk neutral valuation. The main idea of "risk neutral valuation" is that market prices of assets give information on the "perception" of the risk. Give an example.

Assume there are two bonds. The first one is issued by a government : its maturity is one year, it nominal is 100 euros and its yield is $r = 5\%$. The price of this bond is hence (by definition of the yield) :

$B_1 = \frac{100}{1+r} = \frac{100}{1,05}$.

The second one is a corporate bond issued by a firm F, its repay is also 100 euro in one year, but there is a risk of default. Assume for instance that there is some probability P_d that the repay will be only 80. Its nominal yield is hence larger than the risk-free bond because investors demand a better return to compensate this risk of default. Say R =10% yield for this second bond.

The price of this second bond is hence : $B_2 = \frac{100}{1+R} = \frac{100}{1,1}$. Note that the expected present value is $E_2 = \frac{80P_d+100(1-p_d)}{1,05}$.

Assume that on top of these two bonds, there is a new financial product : a CDS (credit default swap). This product promises 1 euro in case of default of the firm F. In some sense, this is an insurance product because it indemifies loss due to default. Consider the following portfolio strategy : buy the second bond and immunate risk by buying 20 CDS. This strategy gives exactly 100 euros in one year, and hence completely identical with the bond 1. The price of this portfolio must then be equal to $B1$ (we say that the market is arbitrage free).

$$B_2 + 20P = B_1$$

This gives the price of the CDS :

$$P = \frac{B_1 - B_2}{20} = \frac{100}{20}\left(\frac{1}{1+r} - \frac{1}{1+R}\right)$$

The first miracle is that the price of the CDS (which is an insurance product) can be computed without knowing the true probability of default, as soon as we know the prices of the underlying asset and the risk free rate (government bond yield)! This property is the basis of what is called "risk neutral valuation" and is used thouroughly to price derivatives and other "structured" products.

At this point it is interesting to compute the expected present value of the CDS : $E_{CDS} = \frac{P_d}{1+r}$. It is easy to show that there is equality of price P and expected present value E_{CDS}, $P = E_{CDS}$, if and only if $B_2 = E_2$. In fact one can show (see the next example) that risk-aversion of investors implies that $B_2 \leq E_2$: they demand a larger expected return to compensate the risk taken, the difference is what is called the risk premium.

Notice that we can define a fictitious probability of default $\widehat{P}_d$, as the probability such that the price of the CDS, is equal to its expected present value :

$$\widehat{P}_d = (1+r)P$$

So that :

$$(1+r)P = \widehat{P}_d = (1+r)\left(\frac{B_1 - B_2}{20}\right) = \frac{100}{20}\left(1 - \frac{1+r}{1+R}\right)$$

The point is then that the expected present value of the bond computed with this probability is also equal to its price!

$$\frac{\widehat{P}_d 80 + (1-\widehat{P}_d)100}{1+r} = \frac{100 - \widehat{P}_d 20}{1+r} = \frac{100}{1+r} = B_2$$

We get hence the interesting result : the prices of the three assets are equal to the expected present values computed with a well defined fictitious probability that we call “risk neutral probability”.

Matematically, things appear quite simple. We are in a world where there are only two states of nature : no default of firm F, default of firm F. The repay in one year can be described by a two components vector, each component being the repay in the corresponding state of nature. For the risk-free bond it is $d_1 = (100, 100)$ whereas it is $d_2 = (100, 80)$ for the second bond and $d_{CDS} = (0, 1)$ for the CDS. Clearly these 3 vectors are not independent since one of them is a linear combination of the two other :

$$d_1 = d_2 + 20 d_{CDS}$$

That we can write :

$$d_{CDS} = \frac{1}{20} d_1 - \frac{1}{20} d_2$$

We say that the market is complete in the sense that any asset whose repay is $d = (a, b)$ can be written as a linear combination of the two bonds :

$$d = \frac{1}{20}\left(d_2 - \frac{80}{100} d_1\right) a + \frac{d_1 - d_2}{20} b$$

The “arbitrage free “ hypothesis implies that the price of this new asset is simply the same linear combination of the prices of the two bonds :

$$P = \frac{1}{20}\left(B_2 - \frac{80}{100}B_1\right)a + \frac{B_1 - B_2}{20}b$$

In the formula above, the price appears as a weghted sum of payments. The weights, $\frac{1}{20}\left(B_2 - \frac{80}{100}B_1\right)$ and $\frac{B_1 - B_2}{20}$ are positive. Their sum is (obviously) equal to $\frac{B_1}{100} = \frac{1}{1+r}$. So that the two numbers $\frac{1+r}{20}\left(B_2 - \frac{80}{100}B_1\right)$ and $(1+r)\left(\frac{B_1 - B_2}{20}\right)$ are the risk neutral probabilities of the two events.

We see that this risk neutral probability is only defined with the current prices of the given assets. It is not necessary to know the true probability, provided we know the market prices. As these prices depned on supply and demand, this probability distribution hence also depends on “supply” and “demand” of these assets.

1.4.2. Supply, demand and risk aversion : where the risk neutral probability comes from. Consider a very simple “market” where there are only two individuals A and B. A is an entrepreneur. His activity provides him a random net income (profit). There are two “states of nature” (events) : the good state where the income is large W_H and the bad state where it is low $W_L < W_H$. B is a lady who has a constant income w. In this economy the total income is hence $W_H + w$ or $W_L + w$. A has a real initial position : he is exposed to a risk and would like to mitigate it, that is to insure his income. What means insure? That means that he would like to have a final income X such that X_H (final income when W_H) is not very different than X_L (final income when W_L). He can try to trade with B by selling him some share of his firm. If he sells a share α at a price p we will have the following final situation

- A will have $X_L = \alpha p + (1-\alpha)W_L$ and $X_H = \alpha p + (1-\alpha)W_H$
- And B $Y_L = w - \alpha p + \alpha W_L$ and $Y_H = w - \alpha p + \alpha W_H$

In somme sense, by doing so, A decreases the risk he is exposed to : the gap between high and low income is $X_H - X_L = (1-\alpha)(W_H - W_L) \leq W_H - W_L$

What are the possible values of p? More precisely, which value of p can be considered as an “equilibrium” value?

First of all we necessarily have $W_L \leq p \leq W_H$.

Indeed $p < W_L$ is surely not accepted by A : for $\alpha \geq 0$ this price would give him an income always lower than W_L ! In the same way, $p > W_H$ cannot be accepted by B

PROPOSITION 5. *if p is an equilibrium price, then* $W_L \leq p \leq W_H$, or equivalently there exists $0 \leq q_L \leq 1$, $0 \leq q_H \leq 1$ / $q_H + q_L = 1$ and $q_H W_H + q_L W_L = p$. Or equivalently :there existsts q_H $0 \leq q_H \leq 1$ /$p = W_L + q_H(W_H - W_L)$

But obviously, there are a lot of such values of p. The equlibrium value of p depend on A and B "behaviour". At which price p A is ready to sell α and B ready to buy α?

At this time of the reasonning, there is an interesting remark. When you look at these two equations, $q_H + q_L = 1$ and $q_H W_H + q_L W_L = p$ where $0 \leq q_L \leq 1$, $0 \leq q_H \leq 1$, you have the idea to replace q_H and q_l by the probabilities π_H and π_L of the two sates of nature. If you do so p is simply equal to the expected value of the firm! Can A and B agree on this particular price?

Look at B. She starts with a sure income w if she buys α share at price $\pi_H W_H + \pi_L W_L$ she will have a random income with an unchanged expected value :

$$\pi_H Y_H + \pi_L Y_L = \pi_H (w - \alpha p + \alpha W_H) + \pi_L (w - \alpha p + \alpha W_L) = w - \alpha p + \alpha (\pi_H W_H + \pi_L W_L) = w$$

At this price, buying increase the risk borne by B without increasing the average income. If she does not like risk, she will not accept, and this price cannot be an equilibrium price.

We feel that we need to model the attitude towards risk of B to model her demand in the risky asset. Intuitively, she will be ready to take risk of her income under the condition that her expected average income goes up. That is under the condition $p \leq \pi_H W_H + \pi_L W_L$.

1.4.3. Risk neutral probability. This allows to identify the risk-neutral probability

PROPOSITION 6. *If agents are risk-averse and if $q_H W_H + q_L W_L = p$ is an equilibrium price, then there exists two numbers $v_L \geq 1 \geq v_H$ such that $q_H = \pi_H v_H$ and $q_L = \pi_L v_L$. q is called the risk neutral probability distribution.*

In order to show the proposition we need to define risk aversion

1.4.4. Risk aversion. How can we model risk aversion? The idea is very simple : one individual is risk-averse if he accepts to take risk only if his average income icreases.

Take an individual with a sure income w_0. We tell him you can stay with this income or take a risk to increase or decrease your income : with a probability $1/2$ you get an extra income of $a + x$ and with probability $1/2$ you get $a - x$, where $a \geq 0$ and $x \geq 0$. The individual has hence the choice beween two "lotteries" :

	$1/2$	$1/2$
lottery S	w_0	w_0
lottery T	$w_0 + a + x$	$w_0 + a - x$

DEFINITION 7. We say that the individual is risk averse if , for all x, there exists a threshold $\underline{a}(x) \geq 0$, increasing with x , such that he prefers the lottery T only if $a \geq a(x)$.

1.4.4.1. *The expected-utility assumption.* One way to capture risk aversion is to assume that the individual values lottery by computing the expected value of som concave function u of the payments :

$$\text{Utility of S} = \frac{1}{2}u(w_0) + \frac{1}{2}u(w_0) = u\left(w_0\right)$$

and

$$\text{Utility of T} = \frac{1}{2}u(w_0 + x + a) + \frac{1}{2}u(w_0 - x + a)$$

If u is continuous, increasing and concave then we get the result : there exists $\underline{a}(x) \geq 0$ such that $a \geq a(x) \Longleftrightarrow$ Utility of A$\leq$ Utility of B.

Indeed : if u is concave then $\frac{1}{2}u(w_0+x+a)+\frac{1}{2}u(w_0-x+a) \leq u\left(\frac{1}{2}(w_0+x+a)+\frac{1}{2}(w_0-x+a)\right) = u(w_0+a)$

For $a = 0$, the utility of T is lower than those of S, for $a = x$, it is larger. The proof that $\underline{a}(x)$ is increasing is let to the reader.

1.4.4.2. *Risk sharing.* Selling shares to B, amounts to risk sharing between A and B. A decreases income risk whereas B increases.

Just look at the values of utilities of the final wealth of our two individuals. For A, the utility if he sells α at the price p is :

$$U_A(p, \alpha) = \pi_H u_A\left(\alpha p + (1-\alpha)W_H\right) + \pi_L u_A\left(\alpha p + (1-\alpha)W_L\right)$$

For B :

$$U_B(p, \alpha) = \pi_H u_A\left(w - \alpha p + \alpha W_H\right) + \pi_L u_A\left(w - \alpha p + \alpha W_L\right)$$

What is the supply function of A? In other words, how many shares is he ready to sell if the price is p? The answer is, the value of α that maximizes U_A for the given value of p.

We have hence, computing the derivative of U_A with respect of α:

$$\pi_H u_A'\left(X_H\right)\left(p - W_H\right) + \pi_L u_A'\left(X_L\right)\left(p - W_L\right) = 0$$

The same reasonning for B :

$$\pi_H u_B'\left(Y_H\right)\left(W_H - p\right) + \pi_L u_B'\left(Y_L\right)\left(W_L - p\right) = 0$$

The final allocation is an equilibrium if the supply is equal to the demand :

CLAIM 8. X_H, X_L, Y_H, Y_l , p and α are at equilibrium if :

$$\begin{cases} \pi_H u_A'(X_H)(p - W_H) + \pi_L u_A'(X_L)(p - W_L) = 0 \\ \pi_H u_B'(Y_H)(W_H - p) + \pi_L u_B'(Y_L)(W_L - p) = 0 \\ X_H + Y_H = w + W_H \\ X_L + Y_L = w + W_L \\ X_H = \alpha p + (1-\alpha) W_H \\ X_L = \alpha p + (1-\alpha) W_L \end{cases}$$

The first equation is quite interesting : we can rewrite it :

$$p = \pi_H \frac{u_A'(X_H)}{\pi_H u_A'(X_H) + \pi_L u_A'(X_L)} W_H + \pi_L \frac{u_A'(X_L)}{\pi_H u_A'(X_H) + \pi_L u_A'(X_L)} W_L$$

The second one is very similar :

$$p = \pi_H \frac{u_B'(Y_H)}{\pi_H u_B'(Y_H) + \pi_L u_B'(Y_L)} W_H + \pi_L \frac{u_B'(Y_L)}{\pi_H u_B'(X_H) + \pi_L u_B'(X_L)} W_L$$

We immediately see that p is somewhere in between W_L and W_H but lower than $\mathrm{E}(W) = \pi_H W_H + \pi_L W_L$ as soon as $X_H \geq X_L$. Indeed in this case, by concavity of u, $u'(X_H) \leq u'(X_L)$.

PROPOSITION 9. *At equilibrium the price is* $p = q_H W_H + q_L W_l$ *with :*

$$q_H = \pi_H \left[\frac{u_A'(X_H)}{\pi_H u_A'(X_H) + \pi_L u_A'(X_L)} \right] = \pi_H \left[\frac{u_B'(Y_H)}{\pi_H u_B'(Y_H) + \pi_L u_B'(Y_L)} \right]$$

$$q_L = \pi_L \left[\frac{u_A'(X_L)}{\pi_H u_A'(X_H) + \pi_L u_A'(X_L)} \right] = \pi_L \left[\frac{u_B'(Y_L)}{\pi_H u_B'(X_H) + \pi_L u_B'(X_L)} \right]$$

One says that the price is equal to the risk-neutral expected value : that is the expected value computed with the risk-free probability. The risk free probability is a reweighted probability. The weight of states of natures with low total income is higher than the weight of states of nature with high total income.

EXERCISE 10. Solve the above model with CARA utility functions : $u_i(s) = -\exp -\rho_i s$.

CHAPTER 2

Static model : arbitrage free condition

The arbitrage free condition will imply that there exists some weight such that the prices of existing assets and the price of any asset that can be built with existing assets (portfolios) is simply equal to the weghted sum of its cash flows! If the market is "complete", that implies that any "cash flow profile" can be obtained through some portfolio. This principle is the basis that we call "risk neutral valuation".

2.1. Mathematical preamble

To derive the main result on no arbitrage theory we need some maths. The following proposition is known as "Convex separation theorme". It tells us that if we consider a point outside a closed convex set, one can separate it from this set by an affine hyperplane.

PROPOSITION 11. *in an Hilbert space H (vect space with a norm coming from a scalar product, and complete), if C is a closed convex subset of H, x in H but not in C, then there exists h in H such that , $\forall y \in C, h.x < h.y$*

PROOF. let $P(x)$ the projection of x on C. $P(x) = \arg\min_{y \in C} \| x - y \|$.

It can be shown that this projection exists and is unique :

- it indeed exists :let $d = \inf_{y \in C} \| x - y \|$; obviously, $d > 0$.

Take a sequence y_n in C such that $d^2 \leq \| x - y_n \|^2 \leq d^2 + 1/n$

we have (by the equality of the median) : $\| x - \frac{(y_n + y_m)}{2} \|^2 + \frac{1}{4} \| y_n - y_m \|^2 = \frac{1}{2} \| y_n - x \|^2 + \frac{1}{2} \| y_m - x \|^2$

$\frac{(y_n + y_m)}{2} \in C$ (convex)

we have hence $\| x - \frac{(y_n + y_m)}{2} \|^2 \geq d^2$,and hence : $\frac{1}{4} \| y_n - y_m \|^2 \leq \frac{1}{2n} + \frac{1}{2m}$.

The sequence y_n is a Cauchy seq and then converges towards y which lies in C (closed) and is such that $\| x - y \| = d$

- Unique : let to the reader

We have :

$\forall y \in C \ (x - P(x)) . (y - P(x)) \leq 0,$

indeed $\| ty+(1-t)P(x)-x \|^2=\| t\,(y-P(x))+P(x)-x \|^2= t^2 \| y-P(x) \|^2 +2t\,(y-P(x))\,.\,(P(x)-x)+\| P(x)-x \|^2$

as $\| ty+(1-t)P(x)-x \|^2 \geq \| P(x)-x \|^2$ (since $P(x)$ is the closest y in C)

we have $t \| y-P(x) \|^2 +2\,(y-P(x))\,.\,(P(x)-x) \geq 0$ for all t in [0,1] and y in C.

And hence $(y-P(x))\,.\,(P(x)-x) \geq 0$

take then $h=(P(x)-x)$we have $\forall y \in C$ $h.(P(x)-y) \leq 0$, that is $h.(h+x-y) \leq 0$,that is $h.y \geq h.x+ \| h \|^2$

that is $h.y > hx$ □

2.2. No arbitrage condition in a static model

2.2.1. Arbitrage. Consider the very simple model with two date : date 0 and date 1. At date one, there are several "states of naure", these states of nature capture the fact that, at date 0, future is not known, but the various possibilities are known. Let E the (finite) set of possible states of nature at date 1.

At date 0 several assets are available. The asset i gives rise to a payment, at date 1, $d_i(e)$ in the state e. Let p_i the price at date 0 of this asset : paying p_i allows to get $d_i(e)$ at date 1 in the state e.

We note $\theta=(\theta_1,\theta_2,...,\theta_N)$ a portfolio, θ_i is the quantity of asset i owned by an investissor.

Definition 12. $\theta \neq 0$ is an arbitrage portfolio if $W_0(\theta) = -\sum_{i=1}^K p_i\theta_i = -p.\theta > 0$ and $W_1(\theta,e) = \sum_{i=1}^K \theta_i d_i(e) = \theta.d(e) \geq 0$ for all e.

In the above definition, W_t is the flow of income at date t. If there exists an arbitrage portfolio, that means that one can make money at date 0 without any counterpart at date 1. One says that an arbitrage portfolio "is a free lunch". The no arbitrage condition implies that there does not exist arbitrage portfolio. If the market is at equilibrium, then there cannot be "free lunch". We will assume, hence, that there does not exist arbitrage portfolios.

A first very clear and obvious consequence of such an assumption is that if two portfolios give the same returns then they must have the same price. Indeed, assume that two portfolios θ and θ' are such that $\theta.d=\theta'.d$ then if we had $p.\theta' < p.\theta$ then $\theta'-\theta$ would be an arbitrage portfolio.

This have a very practical consequence :

Definition 13. "arbitrage pricing", that is computing the price of a given "new" asset, consists in finding a portfolio of the existing assets that gives the same return. On says that the portfolio replicates the asset.

A very important case must be enhanced : the case of complete markets.

DEFINITION 14. The market is complete if any new asset (that is any profile of payments $W(e)$, $e \in E$ can be replicated :

$$\forall W(e),\ \exists \theta \in \mathtt{R}^K, \sum_{i=1}^{K} \theta_i d_i(e) = W(e)$$

In a complete market there must be as many assets as numbers of states of nature : $K \geq |E|$. Moreover the linear application $\theta \longmapsto W_1(\theta)$ mapping $\mathtt{R}^K$ in $\mathtt{R}^{|E|}$ must be surjective.

We have the very important following proposition that characterizes markets where there is no arbitrage possibility.

PROPOSITION 15. *There does not exist arbitrage portfolio if and only if prices and payments are such that there exists a vector of positive numbers $q(e)$ such that :*

$$\forall i,\ p_i = \sum_e q(e) d_i(e)$$

The coefficients $q(e)$ are called state pseudo-prices.

PROOF. It can be shown that the subset C of $\mathtt{R}^K$ of vector such that $v \in C \iff \exists \lambda \in \mathtt{R}^{+|E|}, v_i = \sum_e \lambda(e) d_i(e)$, is a closed (not trivial) convex (trivial) set.

Assume then that $p \notin C$.

That would mean that Proposition 14 does not hold.

If so the convex separation theorem says that there exists h in $\mathtt{R}^K$ such that $\forall v \in C, h.p < h.v$.

At this step it is important to remark that this implies $\forall v \in C, h.v \geq 0$. Indeed if it were not the case there would exist v_0 in C such that $h.v_0 < 0$, as $kv_0 \in C$ for all $k \geq 0$, $h.(kv_0) = k(h.v_0)$ would go to $-\infty$ for k going to $+\infty$ but that woud contradict $h.p < h.(kv_0)$.

Moreover, we have $\inf_C (h.v) = 0$ since $0 \in C$. This implies $h.p < 0$. So h is an arbitrage portfolio. □

Remark that there can exist several $q(e)$ verifying the above equality. But, if there exists a particular asset such that $d = 0$ except for one particular state of nature e' for which it is equal to 1, then its price is $q(e')$. In this case $q(e')$ is the price (the state price) you have to pay to obtain 1 euro only in the state e'. By extension we call state psudo price the coefficient s $q(e)$. When the market is complete, $q(e)$ is the true state price, and hence uniquely defined.

PROPOSITION 16. *If the market is complete, and verifies the no arbritrage condition, then there exists a unique E-uple q such that :*

$$\forall i,\ p_i = \sum_e q(e)d_i(e)$$

2.2.2. Pricing. It is interesting to use the formula above to compute the price of a portfolio. Indeed, we have for a portfolio θ :

$$-W_0(\theta) = \theta.p = \sum_i \theta_i \sum_e q(e)d_i(e) = \sum_e q(e)W_1(\theta, e)$$

That means that the value at date 0 of the portfolio is simply the present discounted value of its payments computed with the weights q.

This formula is the one one uses when a financial product can be obtained by a combination of existing assets (one says duplicated).

When there is a risk-free asset whose yield is r, the no arbitrage condition gives :

$$1 = \sum_e q(e)(1+r)$$

Let then $\pi(e) = (1+r)q(e)$ we have :

Proposition 17. *If there exists a risk-free asset then for any portfolio we have :*

$$-W_0(\theta) = \theta.p = \frac{1}{1+r}\sum_e \pi(e)W_1(\theta, e)$$

π is called a "risk-neutral" probability and the formula above reads : the value of the portfolio is the discounted expected value of its cash flow (under some risk-neutral probability). Note that π is not uniquely defined if the market is not complete.

If the market is complete, then any income flow $W(e)$ at date 1 can be priced.

Proposition 18. *In a complete market with no arbitrage the value at date 0 of an income flow $W_1(e)$ at date 1 writes :*

$$P = \sum_e q(e)W_1(e)$$

If we note $\sum_e q(e) = 1 + r$ the yield of the risk free asset, and $\pi(e) = (1+r)q(e)$ then the formula writes :

$$P = \frac{1}{1+r} \sum_{e} \pi(e) W_1(e) = \frac{1}{1+r} E_\pi \left[W_1\right]$$

One says that the value of an asset is equal to the the expected discounted value of its cash flows computed with the risk-neutral probability.

2.2.3. Example : The model of Cox Ross and Rubinstein with 2 dates. In this model there are two assets and two states of nature $\{u, d\}$. The risk free asset has a yield r. The other asset is risky. Its price at date 0 is S_0 and the cash obtained at date 1 is either low , $S_1(d)$ or high, $S_1(u)$.

The equations giving the state prices are :

$$1 = (1+r)q(u) + (1+r)q(d)$$

$$S_0 = q(u)S_1(u) + q(d)S_1(d)$$

There equations have positive soulutions if and only if :

$$\frac{S_1(d)}{S_0} \leq 1 + r \leq \frac{S_1(u)}{S_0}$$

And the market is complete if $S_1(d) \neq S_1(u)$.

Finally the state prices are :

$$(1+r)q(u) = \frac{1 + r - \frac{S_1(d)}{S_0}}{\frac{S_1(u)}{S_0} - \frac{S_1(d)}{S_0}} = \frac{(1+r)\,S_0 - S_1(d)}{S_1(u) - S_1(d)}$$

$$(1+r)q(d) = \frac{\frac{S_1(u)}{S_0} - (1+r)}{\frac{S_1(u)}{S_0} - \frac{S_1(d)}{S_0}} = \frac{S_1(u) - (1+r)\,S_0}{S_1(u) - S_1(d)}$$

2.2.4. Optimal hedging. Assume a financial institution sells a financial product which promises $W_1(e)$ at date 1. Assume the price is P. What is the optimal strategy (hedging and pricing) such that the bank will be able to meet his commitment. As he receives P he can buy Δ risky assets and invest the remaining $P - \Delta S_0$ on the risk-free asset. This strategy gives :

$$\begin{cases} W_{\triangle P1}(d) = (1+r)\,(P - \triangle S_0) + \Delta S_1(d) \\ W_{\triangle P1}(u) = (1+r)\,(P - \triangle S_0) + \Delta S_1(u) \end{cases}$$

In order to be hedged, we must have $W_1(e) = W_{\triangle P1}(e)$. This gives :

$$\begin{cases} \Delta = \frac{W_1(u)-W_1(d)}{S_1(u)-S_1(d)} \\ P = \frac{1}{1+r}\left(\frac{S_1(u)-(1+r)S_0}{S_1(u)-S_1(d)}W_1(d) + \frac{(1+r)S_0-S_1(d)}{S_1(u)-S_1(d)}W_1(u)\right) \end{cases}$$

The above solution gives the price as expected (as the weighted value of cash) and also the so called Δ hedging. We can remark that we have also :

$$\Delta = \frac{\partial P}{\partial S_0}$$